Small amplitude oscillations of a flexible thin blade in a viscous fluid: Exact analytical solution

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The oscillation of a thin blade immersed in a viscous fluid has received considerable attention recently due to its importance in technological applications such as the atomic force microscope and microelectromechanical systems. In this article, we consider the general case of a flexible thin blade executing spatially varying small amplitude oscillations in a viscous fluid. Exact analytical solutions for the three-dimensional flow field and hydrodynamic load are derived for both normal and torsional oscillations of arbitrary wave number. This contrasts previous investigations that focus exclusively on the complementary rigid-blade problem, which is two-dimensional, and rely on computational techniques. © 2006 American Institute of Physics. [DOI: 10.1063/1.2395967]

I. INTRODUCTION

The oscillation of solid bodies immersed in a viscous fluid has received considerable attention in the literature since the original work of Stokes who considered a sphere and circular cylinder executing small amplitude oscillations. While these latter problems have simple analytical solutions to the linearized Navier-Stokes equation, in general the flow field and hydrodynamic load due to bodies of more complicated shape must be calculated numerically. One pertinent case is the two-dimensional oscillation of a rigid thin blade of infinite length, which has been considered by a number of workers in various contexts. Arguably, one of the most useful and widely used solutions to this problem is the formulation due to Tuck, who used a boundary integral technique to calculate the flow field numerically. This solution and its extension have found particular application in models for the frequency response of cantilever beams immersed in fluids, which are of technological importance in many fields including microelectromechanical systems (MEMS) and atomic force microscopy (AFM).

Since cantilever beams are elastic bodies, they undergo deflections that depend on the spatial coordinates of the beam. Therefore, it is intrinsically important to investigate the case where the thin blade is not rigid, as has been considered in previous work, but executes spatially varying oscillations along its length. This latter problem generates a genuine three-dimensional flow field, and thus cannot be solved by conventional techniques that have been used successfully for two-dimensional and axisymmetric flows. While this problem could be tackled using standard three-dimensional numerical techniques for the linearized Navier-Stokes equation, at present there is no numerical or analytical solution in the literature.

In this article, we derive exact analytical solutions for the three-dimensional flow field and hydrodynamic load on a thin blade executing normal and torsional oscillations with arbitrary wave number (see Fig. 1). This is achieved by using the exact solution to the linearized Navier-Stokes equation for the three-dimensional flow field above a harmonically oscillating half-space and the principle of linear superposition. In so doing, we recover the numerical results obtained by the boundary integral method of Tuck for the singular case of zero wave number, which corresponds to a genuine two-dimensional flow field. For nonzero wave numbers, the Stokes paradox that manifests itself in the two-dimensional flow field disappears in these three-dimensional problems, as expected. Detailed numerical results for the hydrodynamic load as a function of wave number, which are derived from these analytical solutions, are presented. Asymptotic formulas for the hydrodynamic load are also given and used to demonstrate the validity and accuracy of the new exact solutions. The required exact solution for the flow above a half-space, whose surface is executing normal sinusoidal motion with arbitrary spatial periods in two orthogonal directions, is given in the Appendix. Application of these fundamental results to the frequency response of microcantilever beams immersed in viscous fluids, which account for the three-dimensional fluid motion generated by the beam, will be presented elsewhere.

II. THEORY

We consider a thin blade of zero thickness and infinite length oscillating in an unbounded viscous fluid (see Fig. 1). Both normal and torsional oscillations of arbitrary wave number are allowed. As in previous studies, the amplitude of oscillations is assumed much smaller than the characteristic length scales of the flow, so that the Navier-Stokes equation can be linearized. The fluid is taken to be incompressible. Importantly, these assumptions are valid in many cases of practical interest, as discussed in detail in Ref. 10.

The fluid motion is governed by the linearized Navier-Stokes equations, and since we focus on oscillatory motion, it is appropriate to consider the corresponding Fourier transformed equations.
If results in the time domain are required, they can be obtained by taking the inverse Fourier transform of the solutions presented.

As the basis for calculating the flow field around the blade, we consider the flow above an infinite half-space whose surface is executing normal harmonic motion in two orthogonal directions,

\[ Z(x,y|\omega) = Z_0 e^{imx} e^{iny}, \]

where \( m \) and \( n \) are the wave numbers in the \( x \) and \( y \) directions, respectively, \( \omega \) is the radial frequency of oscillation, \( Z_0 \) is the oscillation amplitude, and \( i \) is the usual imaginary unit. The \( z \)-direction is normal to the surface of the half-space. The fluid lies in the region \( z > 0 \) and is unbounded.

The exact solution to the linearized Navier-Stokes equation for this flow problem is given in the Appendix,

\[ u(x,y,z|\omega) = \nabla \times \nabla \times (F(x,y,z|\omega) \hat{k}), \]  
\[ p(x,y,z|\omega) = \frac{\rho \omega^2 Z_0}{\sqrt{m^2 + n^2}} e^{imx} e^{iny} \times \frac{\sqrt{m^2 + n^2 - i\omega}}{\sqrt{m^2 + n^2 - i\omega + \frac{\omega}{\nu}}} e^{-\sqrt{m^2+n^2} z} \]

and \( \nu = \mu / \rho \) is the kinematic viscosity. The flow field and pressure around the thin blade can then be constructed from this fundamental solution using the principle of linear superposition. The individual cases of normal and torsional oscillations, of arbitrary wave number, shall now be considered.

A. Normal modes

First, we consider normal modes [see Fig. 1(a)] with wave number \( k \) whose displacement function in the \( z \) direction along the length of the blade is

\[ w(x,y,0|\omega) = -i \omega Z_0 e^{ikx} : |y| \leq \frac{b}{2}, \]
\[ p(x, y, 0|\omega) = 0 : |y| > \frac{b}{2}. \]  

(7b)

Noting that the boundary condition in Eq. (7) is an even function of the \( y \) coordinate, the general solution to the flow field can be constructed directly from Eqs. (4) and (5) using the principle of linear superposition,

\[
F(\hat{x}, \hat{y}, \hat{z}|\omega) = -i \omega Z_0 e^{i\xi \hat{z}} \int_0^\infty \chi(\lambda, \kappa, \text{Re}) \cos(\lambda \hat{y}) \times \left( \frac{1}{\sqrt{\kappa^2 + \lambda^2} e^{-\sqrt{\kappa^2 + \lambda^2} \hat{z}}} - \frac{1}{\sqrt{\kappa^2 + \lambda^2 - i \text{Re}}} e^{-\sqrt{\kappa^2 + \lambda^2 - i \text{Re}} \hat{z}} \right) d\lambda,
\]

(8a)

and

\[
p(\hat{x}, \hat{y}, \hat{z}|\omega) = -\rho \omega^2 Z_0 e^{i\xi \hat{z}} \int_0^\infty \chi(\lambda, \kappa, \text{Re}) \cos(\lambda \hat{y}) e^{-\sqrt{\kappa^2 + \lambda^2} \hat{z}} d\lambda,
\]

(8b)

where all coordinates have been scaled by the width of the blade such that \( \hat{x} = x/b, \hat{y} = y/b, \hat{z} = z/b \). The Reynolds number \( \text{Re} \) and normalized wave number \( \kappa \) are defined

\[
\text{Re} = \frac{\rho b^2}{\mu}, \quad \kappa = kb,
\]

(9)

respectively, whereas \( \lambda = mb \).

The function \( \chi(\lambda, \kappa, \text{Re}) \) is to be determined by application of the boundary conditions in Eq. (7). Substituting Eqs. (4a), (8a), and (8b) into Eq. (7) then yields the required conditions

\[
\int_0^\infty \chi(\lambda, \kappa, \text{Re}) \sqrt{\kappa^2 + \lambda^2} \left( 1 - \frac{\sqrt{\kappa^2 + \lambda^2}}{\sqrt{\kappa^2 + \lambda^2 - i \text{Re}}} \right) \cos(\lambda \hat{y}) d\lambda = 1 : |\hat{y}| \leq \frac{1}{2},
\]

(10a)

and

\[
\int_0^\infty \chi(\lambda, \kappa, \text{Re}) \cos(\lambda \hat{y}) d\lambda = 0 : |\hat{y}| > \frac{1}{2}.
\]

(10b)

Note that in the limit as \( \text{Re} \to \infty \), the inviscid solution is recovered.

Once the unknown function \( \chi(\lambda, \kappa, \text{Re}) \) has been determined by solving Eq. (10), the hydrodynamic force in the \( \hat{z} \) direction per unit length \( f(\hat{x}|\omega) \) acting on the beam can be calculated by integrating the normal component of the stress tensor over the blade surface. Since the normal to this surface is parallel to its direction of motion, it then follows from the no-slip boundary condition and the equation of continuity that the pressure alone contributes to the total force in the \( \hat{z} \) direction. Integrating the pressure jump between the upper and lower surfaces of the blade then gives

\[
f(\hat{x}|\omega) = 4 \rho \omega^2 b^2 Z_0 e^{i\xi \hat{z}} \int_0^{1/2} \int_0^\infty \chi(\lambda, \kappa, \text{Re}) \cos(\lambda \hat{y}) d\lambda d\hat{y},
\]

(11)

which leads to the required result,

\[
f(\hat{y}|\omega) = 4 \rho \omega^2 b^2 Z_0 e^{i\xi \hat{z}} \int_0^\infty \chi(\lambda, \kappa, \text{Re}) \frac{\sin(\lambda/2)}{\lambda} d\lambda.
\]

(12)

From Eqs. (1) and (6) the general form for the force per unit length can be written as

\[
f(\hat{y}|\omega) = \frac{\pi}{4} \rho \omega^2 b^2 \Gamma(\kappa, \text{Re}) Z_0 e^{i\xi \hat{z}},
\]

(13)

where \( \Gamma(\kappa, \text{Re}) \) is a complex valued dimensionless force per unit length whose real and imaginary components correspond to the inertial and dissipative components of the force, respectively. From Eqs. (12) and (13) we then obtain

\[
\Gamma(\kappa, \text{Re}) = \frac{16}{\pi} \int_0^\infty \chi(\lambda, \kappa, \text{Re}) \frac{\sin(\lambda/2)}{\lambda} d\lambda.
\]

(14)

We now turn our attention to the solution of \( \chi(\lambda, \kappa, \text{Re}) \). Importantly, we expect the pressure to contain a square root singularity near the edges of the blade at \( \hat{y} = \pm 1/2 \). Consequently, the general form chosen for \( \chi(\lambda, \kappa, \text{Re}) \) must be capable of capturing this behavior. With this property in mind, we then represent the function \( \chi(\lambda, \kappa, \text{Re}) \) formally as a linear combination of Bessel functions. Substituting this into Eq. (10b) and using the standard integral properties of Bessel functions then leads to the following general solution that satisfies Eq. (10b):

\[
\chi(\lambda, \kappa, \text{Re}) = \sum_{m=1}^M a_m f_{2m-1} \left( \frac{\lambda}{2} \right),
\]

(15)

where the coefficients \( a_m \) implicitly depend on the normalized wave number \( \kappa \), the Reynolds number \( \text{Re} \), and are to be chosen such that Eq. (10a) is satisfied.

The utility of this general form is immediately visible upon calculation of the pressure, which is obtained by substituting Eq. (15) into (8b). Noting that the pressure between top and bottom surfaces is antisymmetric about \( z = 0 \) leads to the following result for the pressure jump across blade, i.e.,

\[
\Delta p(x, y|\omega) = p(x, y, 0^+|\omega) - p(x, y, 0^-|\omega):
\]

\[
\Delta p(\hat{x}, \hat{y}|\omega) = \rho \omega^2 b Z_0 e^{i\xi \hat{z}} \sum_{m=1}^M a_m \frac{4}{\sqrt{1 - 4 \hat{y}^2}} T_{2m-2}(\sqrt{1 - 4 \hat{y}^2}),
\]

(16)

where \( T_m \) are Chebyshev polynomials of the first kind of order \( m \) (Ref. 21). Since even Chebyshev polynomials are \( O(1) \) at \( \hat{y} = \pm 1/2 \), Eq. (16) is seen to naturally capture the square root singularity in the pressure at \( \hat{y} = \pm 1/2 \). This feature ensures that the solution converges rapidly with increasing \( M \), as will be illustrated in Sec. III.

To evaluate the coefficients \( a_m \), we follow the procedure used in Ref. 20 for the corresponding inviscid problem, and expand Eq. (10a) as a power series \( \hat{y} \). Substituting Eq. (15)
into Eq. (10a), performing this expansion and equating equal powers of \( \hat{y} \), yields the following system of linear equations for the coefficients \( a_m \):

\[
\sum_{m=1}^{M} A_{q,m} a_m = \begin{cases} 
1 : & q = 1, \\
0 : & q > 1,
\end{cases}
\]

(17)

where

\[
A_{q,m} = \int_0^\infty \lambda^{2q-2} \sqrt{\kappa^2 + \lambda^2} \left( 1 - \frac{\sqrt{\kappa^2 + \lambda^2}}{\sqrt{\kappa^2 + \lambda^2} - i \text{Re}} \right) \times J_{2m-2} \left( \frac{\lambda}{2} \right) d\lambda.
\]

(18)

Equation (18) can be evaluated exactly to give

\[
A_{q,m} = A_{q,m}^* + A_{q,m}^{\text{Re}},
\]

(19)

where

\[
A_{q,m}^* = -\frac{4^{q-1}}{\sqrt{\pi}} G_{13}^{21} \left( \frac{\kappa^2}{16} 0 \ m + 1 \ q - m + 1 \right),
\]

(20a)

\[
A_{q,m}^{\text{Re}} = -\kappa^2 \frac{2^{q-5}}{\sqrt{\pi}} G_{13}^{21} \left( \frac{\kappa^2 - i \text{Re}}{16} 0 \ m + 2 \ q - m \right)
- \frac{4^{q-1}}{\sqrt{\pi}} G_{13}^{21} \left( \frac{\kappa^2 - i \text{Re}}{16} 0 \ m + 1 \ q - m + 1 \right),
\]

(20b)

in terms of the Meijer G function.\(^{22}\) The unknown coefficients \( a_m \) are then obtained by taking the inverse of Eq. (17),

\[
a_m = A_{m,1}^{-1},
\]

(21)

where \( A_{m,1}^{-1} \) is the \( m \)th row element of the first column in the inverse matrix of \( A_{q,m} \). Equations (14) and (15) give the required result,

\[
\Gamma_\alpha(\kappa, \text{Re}) = 8a_1.
\]

(22)

The asymptotic limit of \( \kappa \to \infty \) can be readily computed from Eq. (10), which becomes

\[
\int_0^\infty \chi(\kappa, \text{Re}) \cos(\lambda \hat{y}) d\lambda = \frac{1}{\kappa} \frac{\sqrt{\kappa^2 - i \text{Re}}}{\sqrt{\kappa^2 + \lambda^2} - i \text{Re} - \kappa} : |\lambda| \leq \frac{1}{2},
\]

(23a)

\[
\int_0^\infty \chi(\kappa, \text{Re}) \cos(\lambda \hat{y}) d\lambda = 0 : |\lambda| > \frac{1}{2},
\]

(23b)

whose exact solution is

\[
\chi(\kappa, \text{Re}) = \frac{1}{2\kappa} \frac{\sqrt{\kappa^2 - i \text{Re}}}{\sqrt{\kappa^2 - i \text{Re} - \kappa}} J_1 \left( \frac{\lambda}{2} \right) \quad \text{as} \quad \kappa \to \infty.
\]

(24)

From Eqs. (14) and (24) we then obtain

\[
\Gamma_\alpha(\kappa, \text{Re}) = \frac{8}{\pi \kappa} \frac{\sqrt{\kappa^2 - i \text{Re}}}{\sqrt{\kappa^2 + \lambda^2} - i \text{Re} - \kappa} \quad \text{as} \quad \kappa \to \infty,
\]

(25)

a result that is readily verified by solving the two-dimensional problem in this limit using the streamfunction. Importantly, this asymptotic solution is valid for all values of \( \text{Re} \). Note that as \( \text{Re} \to \infty \) within this limit, the inviscid result in Refs. 18 and 19 is recovered, as required.

**B. Torsional modes**

Next, we turn our attention to the complementary problem of torsional oscillation [see Fig. 1(b)], which is solved by extending the analysis presented in the previous section. The angle function along the length of the blade in this case is

\[
\Phi(\lambda | \omega) = \frac{Z_0}{b} e^{i k \lambda},
\]

(26)

where \( Z_0 \) is the amplitude of oscillation at the edge of the beam, i.e., at \( \hat{y} = \pm 1/2 \), and \( k \) is the wave number. This mode of deformation is illustrated in Fig. 1. Invoking the no-slip condition at the surface and noting that the flow has identical symmetry properties to those for the normal modes, gives the following fluid boundary conditions:

\[
w(\hat{x}, \hat{y}, 0 | \omega) = -i \omega Z_0 \hat{y} e^{i k \hat{z}} : |\hat{y}| \leq \frac{1}{2},
\]

(27a)

\[
p(\hat{x}, \hat{y}, 0 | \omega) = 0 : |\hat{y}| > \frac{1}{2}.
\]

(27b)

As for the normal modes, the flow field can be calculated exactly using the general expression given in Eqs. (4) and (5). However, in this case the flow is antisymmetric in the \( y \) coordinate, which leads to

\[
F(\hat{x}, \hat{y}, \hat{z} | \omega) = -i \omega Z_0 e^{i k \hat{z}} \int_0^\infty \xi(\lambda, \kappa, \text{Re}) \sin(\lambda \hat{y})
\times \left( \frac{1}{\sqrt{\kappa^2 + \lambda^2}} e^{-\sqrt{\kappa^2 + \lambda^2} \hat{z}} - \frac{1}{\sqrt{\kappa^2 + \lambda^2} - i \text{Re}} e^{-\sqrt{\kappa^2 + \lambda^2} - i \text{Re} \hat{z}} \right) d\lambda,
\]

(28a)

\[
p(\hat{x}, \hat{y}, \hat{z} | \omega) = -\rho \omega^2 Z_0 e^{i k \hat{z}} \int_0^\infty \xi(\lambda, \kappa, \text{Re}) \sin(\lambda \hat{y}) e^{-\sqrt{\kappa^2 + \lambda^2} \hat{z}} d\lambda,
\]

(28b)

where all nondimensional parameters are as defined in the previous section and \( \xi(\lambda, \kappa, \text{Re}) \) is to be determined. Substituting Eq. (28a) into Eq. (4), and applying the fluid boundary conditions defined in Eqs. (27), then gives the governing equation for \( \xi(\lambda, \kappa, \text{Re}) \),
where \(\kappa\) is a nondimensional complex valued moment per unit length, analogous to \(\Gamma_0^i(\kappa, Re)\) for the normal modes. Calculating the pressure jump between top and bottom surfaces, and using Eq. (30), then gives the required expression for \(\Gamma_0^i(\kappa, Re)\) in terms of \(\zeta(\lambda, \kappa, Re)\),

\[
\Gamma_0^i(\kappa, Re) = \frac{32 \pi}{\lambda} \int_0^\infty \zeta(\lambda, \kappa, Re) \left[ \frac{\sin(\lambda/2)}{\lambda^2} - \frac{\cos(\lambda/2)}{2\lambda} \right] d\lambda.
\]

(31)

To calculate \(\zeta(\lambda, \kappa, Re)\), we choose the ansatz

\[
\zeta(\lambda, \kappa, Re) = \sum_{m=1}^{\infty} b_m J_{2m-1}(\lambda/2),
\]

(32)

that satisfies Eq. (29b) and ensures that the square root singularity at \(\hat{y} = \pm 1/2\) is automatically included. The coefficients \(b_m\) are again to be evaluated such that Eq. (29a) is satisfied. The pressure jump across blade, i.e., \(\Delta p(x, y|Re) = p(x, y|0|Re) - p(x, y|0^+|Re)\), in this case is given by

\[
\Delta p(\hat{x}, \hat{y}|Re) = \rho_0 2^3 B q e^{i2\kappa \pi} \sum_{m=1}^{\infty} b_m \frac{8\hat{y}}{\sqrt{1 - 4\hat{y}^2}} U_{2m-2}(\sqrt{1 - 4\hat{y}^2}),
\]

(33)

where \(U_m\) are Chebyshev polynomials of the second kind of order \(m\) (Ref. 21), showing the expected square root singularity at \(\hat{y} = \pm 1/2\).

To determine the coefficients \(b_m\), we substitute Eq. (32) into Eq. (29a) and expand \(\sin(\lambda \hat{y})\) in its power series about \(\hat{y} = 0\), which has an infinite radius of convergence. This gives the following system of linear equations:

\[
\sum_{m=1}^{\infty} B_{q,m} b_m = \begin{cases} \frac{1}{\lambda} : q = 1, \\ 0 : q > 1, \end{cases}
\]

(34)

where the matrix elements \(B_{q,m}\) are given by

\[
B_{q,m} = b^k_{q,m} + b^{Re}_{q,m},
\]

(35)

where

\[
\begin{align*}
B^k_{q,m} &= -\frac{4^2q}{\sqrt{\pi}} G_{13}^{21} \left( \frac{k^4}{16} 0 q + m \frac{3}{2} q - m + 1 \right), \\
B^{Re}_{q,m} &= -\frac{4^{2q-3}}{\sqrt{\pi}} G_{13}^{21} \left( \frac{k^2 - i Re}{16} 0 q + m - 1 \frac{1}{2} q - m + 1 \right) - \frac{4^{2q+1}}{\sqrt{\pi}} G_{13}^{21} \left( \frac{k^2 - i Re}{16} 0 q + m \frac{1}{2} q - m + 1 \right),
\end{align*}
\]

(36a, 36b)

in terms of the Meijer G function. Substituting Eq. (32) into Eq. (31) then gives the required result,

\[
\Gamma_0^i(\kappa, Re) = 4b_1,
\]

(37)

Thus we again observe that the nondimensional moment per unit length \(\Gamma_0^i(\kappa, Re)\) depends only on the first coefficient in the expansion equation (32).

Finally, we examine the asymptotic behavior of \(\Gamma_0^i(\kappa, Re)\). In the limit as \(Re \rightarrow 0\) and \(\kappa \rightarrow 0\), Eq. (29a) reduces to

\[
\int_0^\infty \zeta(\lambda, \kappa, Re) \sin(\lambda \hat{y}) d\lambda = \frac{2i}{Re} \hat{y} : |\hat{y}| \leq \frac{1}{2},
\]

(38)

whose exact solution is given by the first term in the ansatz, Eq. (32), which gives

\[
\zeta(\lambda, \kappa, Re) = \frac{i}{Re} J_1 \left( \frac{\lambda}{2} \right).
\]

(39)

Substituting Eq. (39) into Eq. (31) then gives

\[
\Gamma_0^i(\kappa, Re) = \frac{4i}{Re} \quad \text{as} \quad Re \rightarrow 0 \text{ and } \kappa \rightarrow 0,
\]

(40)

which agrees with the asymptotic expression obtained by Green and Sader by independent means.

In the complementary limit as \(\kappa \rightarrow \infty\), Eq. (29a) becomes

\[
\int_0^\infty \zeta(\lambda, \kappa, Re) \sin(\lambda \hat{y}) d\lambda = \frac{\hat{y} \sqrt{\kappa^2 - i Re}}{\kappa \sqrt{\kappa^2 - i Re - \kappa}} : |\hat{y}| \leq \frac{1}{2},
\]

(41)

Noting that Eq. (41) is identical in form to Eq. (38) enables the exact solution to Eq. (41) to be obtained trivially. This leads to the required expression that is valid for all Re,

\[
\Gamma_0^i(\kappa, Re) = \frac{4}{3\pi k} \frac{\sqrt{k^2 - i Re}}{k \sqrt{k^2 - i Re - \kappa}} \quad \text{as} \quad \kappa \rightarrow \infty,
\]

(42)

a solution that is readily verified by solving the two-dimensional flow problem in this limit using the streamfunction. Also note that in the limit as Re \(\rightarrow \infty\), the inviscid solution presented in Ref. 20 is recovered, as required.

To facilitate implementation, a summary of the key results derived in this section for both the normal and torsional modes is given in Table I. It is important to note that the normalized hydrodynamic loads in both cases depend only on the entries in the first row and first column of the inverse
of the matrices $A_{q,m}$ and $B_{q,m}$. The present formulation naturally accounts for the expected square root singularity at the edges $\hat{y}=\pm 1/2$, and enables the analytical evaluation of the three-dimensional flow field and hydrodynamic load.

### III. RESULTS AND DISCUSSION

We now examine the convergence of the exact analytical solutions, Eqs. (22) and (37), presented in the previous section with respect to the number of terms $M$. It is important to emphasize that the matrix elements $A_{q,m}$ and $B_{q,m}$ depend on $\kappa$ and Re only. Thus, if these matrices are evaluated for $M$ greater than one, at fixed $\kappa$ and Re, then the solutions for all $M$ less than this value can be evaluated at once. This enables the convergence of solutions to be determined easily. Note that both solutions are formally exact in the limit as $M \to \infty$, where the power series expansion for the trigonometric functions used is exact. All computations were performed in Mathematica®5.2, where the Meijer $G$ function is standard.

To begin, we consider the case of $\kappa=0$, corresponding to the two-dimensional flow induced by an oscillating rigid thin blade; this problem was solved independently by Tuck using a boundary integral formulation.\(^9\) We focus our convergence study on the imaginary component of the hydrodynamic load, since this becomes very small for large Re. Results for the minimum value of $\kappa$ required to achieve 99% convergence in the numerical results for the normal mode are given in Fig. 2(a), and denoted $M_{\text{critical}}$. Similar results are obtained for the torsional mode. From Fig. 2(a), it is clear that $M_{\text{critical}}$ depends strongly on the Reynolds number Re, with $M_{\text{critical}}$ increasing notably for Re $> 1$. For Re $< 100$ only a few terms are required to achieve convergence, while for Re $> 100$ significantly more terms are required. As we shall see below, this increase is due to the rapidly varying flow field near the edges $\hat{y}=\pm 1/2$ for large Re.

In Fig. 2(b), we present complementary results for both normal and torsional modes showing the effect of varying both $\kappa$ and Re on the convergence. Results are presented collectively for both the normal and torsional modes with $\kappa=0, 1, 2, 3, 4, 5$. We emphasize that the convergence results for normal and torsional modes are nearly identical for all values of $\kappa$. The results presented indicate that above a critical value of Re, given approximately by $Re_{\text{critical}} \sim \kappa^2$, the value of $M_{\text{critical}}$ is independent of $\kappa$. Moreover, in this regime ($\text{Re} \gg \kappa^2$) the results appear to follow the empirical scaling law $M_{\text{critical}} \sim 1/3\sqrt{\text{Re}}$ for both normal and torsional modes [see Fig. 2(b)]. For $\text{Re} < \kappa^2$, $M_{\text{critical}}$ depends only on $\kappa$ and is independent of Re. In this latter regime, the formula $M_{\text{critical}} \sim 1+1/\sqrt{\kappa}$ is found to give a good empirical fit to the data presented for both normal and torsional modes. Thus, we can formulate the following composite expression:

$$M_{\text{critical}} \sim \max \left(1 + \sqrt{\frac{1}{3} \frac{1}{\kappa}} \frac{1}{\sqrt{\text{Re}}} \right), \quad (43)$$

which can be used to estimate the minimum number of terms $M$ required for convergence. We emphasize, however, that this expression has been derived empirically and thus should be used only as a guide.

We now investigate the validity of the new analytical solutions, Eqs. (22) and (37), by directly comparing them to previous numerical results and asymptotic formulas for the hydrodynamic load. In all cases, $M=20$ terms were used in the exact solutions. First, we consider the limiting case of $\kappa=0$, which corresponds to the two-dimensional rigid-blade problem. In Fig. 3, we show a comparison of the new solutions to numerical results obtained using the boundary integral formulation of Tuck.\(^9\) As is evident from this comparison, the new analytical solutions give excellent agreement.
with the numerical results of Tuck\textsuperscript{9} for both normal and torsional modes with results spanning several orders of magnitude in both real and imaginary components.

Next we consider the opposite limit of $\kappa \gg 1$, where the flow field is also two dimensional but in a direction perpendicular to the $\kappa=0$ case. Here, the asymptotic solutions are given in Eqs. (25) and (42) for the normal and torsional modes, respectively. A comparison of these limiting solutions to the exact solutions is given in Fig. 4 for $\kappa=10$. Note that solutions for both normal and torsional modes differ significantly from the complementary $\kappa=0$ case (cf. Figs. 3 and 4). The asymptotic solution for the normal mode in this $\kappa \gg 1$ limit accurately captures the behavior calculated from the exact solution. Interestingly, a lesser degree of agreement is found for the torsional mode, although the overall behavior is predicted well. This difference in convergence to the $\kappa \gg 1$ asymptotic limit was also observed in the inviscid case,\textsuperscript{20} with the torsional mode requiring a larger numerical value of $\kappa$ to reach this asymptotic limit.

From Eqs. (25) and (42), we find that for fixed $Re$ the real component of the hydrodynamic loads approaches values that are independent of $Re$ in the $\kappa \gg 1$ limit, whereas the imaginary components are proportional to $Re^{-1}$, namely,

$$\Gamma_n(\kappa,Re) = \frac{12}{\pi \kappa} + i \frac{16 \kappa}{\pi Re} \quad \text{as} \quad \kappa \to \infty,$$

$$\Gamma_t(\kappa,Re) = \frac{2}{\pi \kappa} + i \frac{8 \kappa}{3 \pi Re} \quad \text{as} \quad \kappa \to \infty,$$

a feature that is mirrored in results obtained using the exact solutions in Fig. 4.

With the validity of the exact solutions established by comparison with asymptotic and independent numerical results, we turn our attention to investigating the effect of varying normalized wave number $\kappa$ on the hydrodynamic loads. Results of this comparison for the normal and torsional modes are given in Fig. 5. The imaginary (dissipative) component of the load has been multiplied by the Reynolds number $Re$ to facilitate investigation of the Stokes paradox. Note that for $\kappa=0$, these two-dimensional solutions exhibit features of the well-known Stokes paradox, with the hydrodynamic load diverging as $Re \to 0$. This is particularly pronounced for the normal modes, whereas for the torsional mode the logarithmic divergence occurs at second order\textsuperscript{13} and thus is not immediately apparent in the numerical results. However, for nonzero and finite $\kappa$, the flow becomes genuinely three dimensional and it is strikingly evident from Fig. 5 that this manifestation of the Stokes paradox (for two-dimensional flows) disappears in both real and imaginary components of the hydrodynamic load. Note that while the flow is two dimensional for $\kappa \to \infty$, features of the Stokes paradox are not manifest in the solution, as is clear from Eq. (44).

Most significantly from Fig. 5, it is observed that the true hydrodynamic load is well approximated by the solution in the Stokes limit ($Re \to 0$) for all $Re < \kappa^2$, at which point the viscous boundary layer thickness at the solid surface becomes comparable to or greater than the spatial wavelength along the length of the blade. While the position of this turnover point ($Re \sim \kappa^2$) is to be expected, since the viscous flow field can sense the three-dimensional nature of the solid sur-
face at this point, the observation that the imaginary component of the hydrodynamic load (dissipative part) for $Re \ll \kappa^2$ is independent of inertial forces in the fluid, regardless of the value of $\kappa$, may appear somewhat surprising. This behavior can be explained by the choice of length scale for $\kappa$, since for such situations the dominant length scale in the flow is the spatial wavelength $k^{-1}$ rather than the width $b$ of the blade as has been assumed in $Re$.

From Fig. 5, we note that as the wave number $\kappa$ increases, the inertial component of the load (real part) decreases for all values of $Re$. This finding is identical to that for inviscid flow,\cite{20} where it was found that fluid loading approaches zero as $\kappa \rightarrow \infty$, and is in agreement with the asymptotic solutions given in Eq. (44). While the inertial component decreases with increasing $\kappa$, the dissipative load (imaginary part) is found to have more complex behavior.

For $Re \ll \kappa^2$, the dissipative load increases with increasing $\kappa$, in line with the asymptotic expressions in Eq. (44). However, for $Re \gg \kappa^2$, the dissipative load decreases with increasing $\kappa$. This finding establishes that for fixed $Re$, inertial loads approach zero while dissipative loads increase unboundedly as $\kappa \rightarrow \infty$.

Finally, we examine the pressure distributions that give rise to these hydrodynamic loads. In Fig. 6, we present results for the pressure jumps across the blade for both normal and torsional modes, as a function of both wave number $\kappa$ and Reynolds number $Re$. We focus on the real component of the pressure, since this can be compared to the inviscid solution in the limit of large $Re$. The presence of the square root singularity at $\hat{y}=\pm 1/2$ in the pressure distributions is clearly evident in all results for this viscous flow problem. This is to be compared to the inviscid solution, which is zero

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig4.png}
\caption{Normalized hydrodynamic loads for high wave number of $\kappa=10$. Dots are obtained from new exact analytical solutions; solid lines correspond to asymptotic formulas in Eqs. (25) and (42). (a) Normal mode (real component); (b) normal mode (imaginary component); (c) torsional mode (real component); (d) torsional mode (imaginary component).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig5.png}
\caption{Normalized hydrodynamic loads for varying wave number $\kappa=0,1,2,3,5,7,10$, obtained using new exact analytical solutions. (a) Normal mode (real component); (b) normal mode (imaginary component); (c) torsional mode (real component); (d) torsional mode (imaginary component). The dashed line corresponds to the two-dimensional rigid-blade case ($\kappa=0$). Imaginary components are multiplied by $Re$.}
\end{figure}
at $\hat{y} = \pm 1/2$. Nonetheless, away from these edges the viscous solutions approach the inviscid results as Re increases, as required. Interestingly, the position $\hat{y}$, where the pressure turns towards the square root singularity, moves towards the edge $\hat{y} = \pm 1/2$ as Re increases. This increasingly rapid variation in the pressure for larger Re necessitates a commensurate increase in the minimum number of terms $M_{\text{critical}}$ required for convergence of the solution, as observed above.

The results for $\kappa = 0$ are identical to those produced by Tuck.9 Significantly, as the wave number $\kappa$ increases, the pressure profiles become increasingly linear near the middle of the blade in both inviscid and viscous solutions. This behavior is also expected, since the flow becomes two dimensional along the width of the blade in the limit as $\kappa \to \infty$. It is interesting to note that the solutions for Re=10 and 30 are almost indistinguishable for $\kappa = 10$ in both the normal and torsional modes. This behavior is consistent with the results for the normalized hydrodynamic loads in Fig. 5, whose real components are independent of Re for Re $< \kappa^2$. We also observe that the magnitude of the real component of the pressure decreases with increasing wave number $\kappa$, which causes a reduction in the inertial component of the load, as seen in Fig. 5.

IV. CONCLUSIONS

We have presented exact solutions for the three-dimensional flow generated by an oscillating thin blade in a viscous fluid. Both normal and torsional modes of arbitrary wave number were considered. These results extend previous solutions for the two-dimensional flow generated by an oscillating rigid blade9 to the three-dimensional oscillation of a flexible blade and provide analytical solutions in all cases. A summary of the key analytical formulas for calculating the hydrodynamic load is presented in Table I.

It was found that manifestations of the Stokes paradox in the two-dimensional rigid-blade problem disappear in genuinely three-dimensional flows, as expected. Interestingly, however, for cases where the viscous boundary layer thickness is larger than the spatial wavelength of the mode, the (quasistatic) Stokes equations yield the dissipative component of the hydrodynamic load, regardless of the magnitude of Re.

To complete the investigation, asymptotic formulas were also presented and used to verify the exact solutions. The results of this study are expected to be particularly relevant to studies in MEMS and AFM, where thin blades form the basis for many devices and applications.

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APPENDIX: OSCILLATIONS OF A HALF-SPACE

In this appendix, we consider the oscillation of the surface of a half-space in contact with a viscous fluid. The surface executes small normal oscillations with arbitrary periods in two orthogonal directions such that the displacement at the surface is

\[ Z(x, y|\omega) = Z_0 e^{imx} e^{iny}, \tag{A1} \]

where \( m \) and \( n \) are the wave numbers in the \( x \) and \( y \) directions, respectively, \( \omega \) is the radial frequency of oscillation, \( Z_0 \) is the oscillation amplitude, and \( i \) is the usual imaginary unit. The \( z \) direction is normal to the surface of the half-space and the fluid lies in the region \( z > 0 \) and is unbounded. As such, the flow field must vanish an infinite distance from the surface.

The governing equation for the flow is the linearized Navier-Stokes equation, Eq. (1). To solve for the flow field we take the curl of the momentum equation twice, from which we obtain

\[
\nabla^2 \left( \nabla u + \frac{i\omega}{\nu} u \right) = 0. \tag{A2}
\]

Following from Eq. (A1), we then expand the velocity field as

\[ u(x, y, z|\omega) = U(z|\omega) e^{imx} e^{iny}. \tag{A3} \]

Substituting Eq. (A3) into Eq. (A2) gives three identical fourth-order ordinary differential equations for the components of \( U \), whose general solutions are given as linear combinations of the functions

\[
e^{-\sqrt{m^2+n^2} z}, \quad e^{-\sqrt{m^2+n^2-i\omega/\nu} z}. \tag{A4}\]

Applying the usual no-slip condition at the surface, leads to the following fluid velocity boundary conditions:

\[ U(0|\omega) = V(0|\omega) = 0, \quad W(0|\omega) = -i\omega Z_0, \tag{A5} \]

where \( U, V, W \) are the \( x, y, z \) components of the vector field \( U \), respectively. To complete the solution, we then ensure that the original momentum and the continuity equations are satisfied. This leads to the required solution for the components of the vector field \( U \),

\[ U(z|\omega) = \omega Z_0 \sqrt{m^2 + n^2 - i\omega/\nu} \left[ e^{-\sqrt{m^2+n^2} z} - e^{-\sqrt{m^2+n^2-i\omega/\nu} z} \right], \tag{A6a} \]

\[ V(z|\omega) = \omega Z_0 \sqrt{n^2 + m^2 - i\omega/\nu} \left[ e^{-\sqrt{n^2+m^2} z} - e^{-\sqrt{n^2+m^2-i\omega/\nu} z} \right], \tag{A6b} \]

\[ W(z|\omega) = -i\omega Z_0 \sqrt{m^2 + n^2 - i\omega/\nu} \left[ e^{-\sqrt{m^2+n^2} z} - e^{-\sqrt{m^2+n^2-i\omega/\nu} z} \right]. \tag{A6c} \]

The complete velocity field can then be written in the compact notation

\[ u(x, y, z|\omega) = \nabla \times \nabla \times (F(x, y, z|\omega) \hat{k}), \tag{A7} \]

where

\[ F(x, y, z|\omega) = \frac{i\omega Z_0 e^{imx} e^{iny}}{\sqrt{m^2 + n^2 - m^2 + n^2 - i\omega/\nu}} \left[ \sqrt{m^2 + n^2 - i\omega/\nu} e^{-\sqrt{m^2+n^2} z} - \frac{1}{\sqrt{m^2 + n^2}} e^{-\sqrt{m^2+n^2-i\omega/\nu} z} \right], \tag{A8} \]

and \( \hat{k} \) is the unit vector in the \( z \) direction. The pressure field immediately follows from the momentum equation and is given by
The above solution is used as the basis for solving the flow field around the oscillating thin blade in Sec. II.


16This definition of the Reynolds number differs from Refs. 10–15 by a factor of 4, and is preferred in the present context since it appears naturally in the final exact solutions [see Eqs. (22) and (37)]. The convention adopted for the Reynolds number also conforms with Ref. 16. The Reynolds number is often associated with the nonlinear convective inertial term in the Navier-Stokes equation. This latter convention has not been adopted here.


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