Parallel beam approximation for V-shaped atomic force microscope cantilevers

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Due to its simplicity, the parallel beam approximation (PBA) is commonly used in the analytical evaluation of the spring constant of V-shaped atomic force microscope (AFM) cantilevers. However, the point of contention regarding the validity of the PBA is as yet an unresolved issue, which has been exacerbated by some recent contradictory reports. In this paper, we present a detailed investigation of the deflection properties of the V-shaped AFM cantilever, and in so doing, show that the PBA is in fact a valid and accurate approximation, provided the width and length of the parallel rectangular arms are chosen appropriately. As a direct consequence of this finding, we obtain exceedingly simple yet accurate formulas for the V-shaped cantilever, which will be of value to the users of the AFM. © 1995 American Institute of Physics.

I. INTRODUCTION

The knowledge of the deflection properties of atomic force microscope (AFM) cantilevers, in particular, their spring constants, is of fundamental importance to the users of the AFM, especially in the measurement of forces. For the V-shaped cantilever, the most popular, conventional, and simplistic approach to the analytical evaluation of its normal spring constant is the use of the parallel beam approximation (PBA). At this stage we note that an accurate analytical solution to the spring constant of V-shaped cantilevers recently appeared in the literature, however, its form is considerably more complex than that of the PBA. To our knowledge, the PBA was first proposed by Albrecht et al. The fundamental principle of the PBA is as follows: (a) First, the basic assumption is made that the V-shaped cantilever can be approximated by two rectangular cantilevers joined in parallel; (b) then, by using the well-known analytical result for the rectangular cantilever, the spring constant of the V-shaped cantilever can be trivially determined. However, the basic assumption of the PBA has never been formally investigated in a theoretical manner, in order to establish its validity. Furthermore, a number of contradictory reports regarding the accuracy of the PBA have recently appeared in the literature, thus exacerbating the issue.

In this paper, we reexamine the PBA by presenting a detailed analysis of the V-shaped cantilever. Consequently, we show that the V-shaped cantilever can in fact be approximated by two rectangular plates joined in parallel, provided the width and length of the rectangular arms are chosen appropriately, thus validating the basic assumption of the PBA. From this important finding, we then obtain an exceedingly simple expression for the spring constant of V-shaped cantilevers, which displays very good accuracy and therefore we believe will be of considerable practical value to the users of the AFM. We also show that the variation in the spring constant with off-end loading (i.e., the load is placed a finite distance back from the end tip) may be very well approximated by the simple expression for the rectangular cantilever. Furthermore, we present a detailed discussion of the PBA, and in so doing clarify the apparently contradictory reports of Sader et al. and Butt et al. in regard to the accuracy and validity of the PBA.

II. ANALYSIS

We begin by examining in detail the geometry of the V-shaped cantilever, which is illustrated in Fig. 1. As is clear from Fig. 1, the cantilever consists of two skewed rectangular arms and a triangular end piece. We shall commence the analysis by considering the two skewed rectangular arms.

Due to the obvious symmetry of the V-shaped cantilever, we clearly need only consider the analysis of one of the skewed rectangular arms; see Fig. 2. For the purpose of analysis, the skewed rectangular arm is divided into three regions: regions A, B, and C, as illustrated in Fig. 2. Region B is clearly amendable to a beam-type analysis, whose deflection function is primarily in the X direction; see Fig. 2. This is of course provided the skewed aspect ratio of the arm $\lambda = d/L_1$ is far less than unity. Even if this is not strictly correct, it shall be assumed that the analysis is valid to leading order. Our rationale for this comes from the related problem of the analysis of an unskewed square cantilever plate where a beam-type analysis displays an accuracy of ~5%, even though the aspect ratio is unity.

The boundary conditions at the ends of the skewed rectangular arm shall now be examined, i.e., at $x = 0$ and $x = L_1$. At $x = 0$, the usual clamped boundary conditions apply, namely,

$$\left[ w - \frac{\partial w}{\partial x} \right]_{x=0} = 0, \quad n = 1, 2, 3, ..., \quad (1a)$$

$$\frac{\partial^2 w}{\partial y^2} \bigg|_{x=0} = 0, \quad n = 1, 2, 3, ..., \quad (1b)$$

where $x$ and $y$ is the coordinate system presented in Fig. 2 and $w$ is the deflection function.

To leading order, we shall then assume that there is no anticlastic curvature present in the deflection function of the triangular end piece of the V-shaped cantilever, i.e., its deflection function is independent of $y$ (see Fig. 1). Our rationale for this comes from the results for a triangular cantilever...
FIG. 1. Cross-sectional view of V-shaped AFM cantilever showing dimensions and position of end-tip loading \( F \) (displayed as a solid circle). The dashed line \( O-O \) is where the clamped boundary condition is applied.

plate. Thus, for a V-shaped cantilever which is loaded with a normal force within its triangular end piece, clearly a transverse shearing force \( F \), and a bending moment \( M \), will act on the skewed rectangular arms at \( x=L \), in the \( x \) direction. Note that the quantification of \( M \), and \( F \), is not required in the analysis, only a statement of their existence. This therefore specifies the boundary conditions at \( x=L \).

With the boundary conditions specified, we now solve for the deflection function of the skewed rectangular arm. Beginning with region B, it is clear from the above discussion and the usual governing beam equation, whose deflection direction is \( X \) (see Fig. 2), that the general expression for the deflection function in region B is

\[
w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \tag{2}
\]

where the coefficients \( a_0, a_1, a_2, \) and \( a_3 \) are yet to be determined.

At this stage we note that a rigorous analysis of the torsional effects in regions A and C of the skewed rectangular arm is hardly justifiable, given the fact that the boundary conditions and solutions already specified are approximate in form, as discussed above. Therefore, we shall neglect these torsional effects, and hence match the solution in region B directly to the boundary conditions at \( x=0 \) and \( x=L_1 \). We note that the boundary conditions presented above are with respect to the unrotated coordinate system \((x,y)\). However, the deflection function for region B, as specified in Eq. (2), is in terms of the rotated coordinate system \((\tilde{x},\tilde{y})\). We therefore apply the usual rotation of coordinate system transformation to the above specified boundary conditions, from which it can be easily shown that the components of the bending moment \( M_1 \) and transverse shearing force \( F_1 \) in the \( \tilde{x} \) direction are \( M_1 \cos^2 \theta \) and \( F_1 \cos^3 \theta \), respectively, where \( \theta \) is the angle of skew of the rectangular arm, as illustrated in Fig. 2.

Therefore, from the above discussion, it is clear that the approximate boundary conditions for Eq. (2) are

\[
\begin{align*}
\left[ w = \frac{\partial w}{\partial \tilde{x}} \right]_{\tilde{x}=0} &= 0, \\
\left[ EI \frac{\partial^2 w}{\partial \tilde{x}^2} \right]_{\tilde{x}=L_1} &= M_1 \cos^2 \theta, \\
\left[ \frac{\partial}{\partial \tilde{x}} \left( EI \frac{\partial w}{\partial \tilde{x}} \right) \right]_{\tilde{x}=L_1} &= F_1 \cos^3 \theta,
\end{align*}
\]

where \( E \) is Young’s modulus and the moment of inertia \( I \) of the bending cross section is that of region B, namely, \( I = \frac{d^3}{12} \), where \( t \) is the thickness of the plate and \( d \) is the shortest width of the rectangular arm, as illustrated in Fig. 2.

Applying the boundary conditions in Eq. (3) to Eq. (2), we obtain the following solution for the deflection function of the skewed rectangular plate:

\[
w(\tilde{x}) = \frac{F_1 \cos^3 \theta}{6EI} \tilde{x}^3 + \frac{M_1 - F_1 L_1 \cos^2 \theta}{2EI} \tilde{x}, \tag{4}
\]

Furthermore, upon defining \( Z = \tilde{x} \cos \theta \), we immediately see that Eq. (4) becomes

\[
w(Z) = \frac{F_1}{6EI} Z^3 + \frac{M_1 - F_1 L_1}{2EI} Z^2, \tag{5}
\]

where \( Z \) is in fact the deflection function of an unskewed \((\theta=0)\) rectangular plate of length \( L_1 \) and of width \( \tilde{d} \).

Therefore, we have shown that, to leading order, the skewed rectangular plate under consideration possesses an identical deflection function in the \( x \) direction to that of an unskewed rectangular plate of length \( L_1 \) and width \( \tilde{d} \). However, we have considered only one of the skewed rectangular arms of the V-shaped cantilever. By symmetry, it is clear from the above discussion that, to leading order, the two skewed rectangular arms are equivalent to a single unskewed rectangular plate of length \( L_1 \) and width \( 2\tilde{d} \), thus validating the basic assumption of the PBA.

From the above discussion it is clear that, to leading order, the original V-shaped cantilever is equivalent to a cantilever which has its skewed rectangular arms replaced by a single unskewed rectangular plate of length \( L_1 \) and width \( 2\tilde{d} \), as is illustrated in Fig. 3. Note that unlike previous PBA
formulations, we retain the triangular end piece, since its inclusion in the analysis is trivial. To obtain the spring constant \( k \) (defined to be the force required per unit length deflection at the end tip) of the equivalent cantilever, we implement the zeroth-order method presented in Ref. 4, from which we obtain the new PBA result,

\[
k = \frac{E t^2 d}{2 L^2} \cos \theta \left( 1 + \frac{4 d^3}{D^3} (3 \cos \theta - 2) \right)^{-1}
\]

(6)

We emphasize that Eq. (6) is valid only when the load is applied to the end tip of the cantilever. However, in practice the load may be applied a finite distance \( \Delta L \) back from the end tip. This off-end loading can result in a large increase in the spring constant. For the rectangular cantilever, an exceedingly simple expression exists for the variation in the spring constant with off-end loading. Furthermore, it can be shown (see the Appendix) that, to leading order, the variation in the spring constant with off-end loading for the V-shaped cantilever is in fact identical to that of the rectangular cantilever, namely,

\[
K = \frac{k_{\Delta L=\Delta L}}{k_{\Delta L=0}} = \left( \frac{L}{L-\Delta L} \right)^3,
\]

(7)

where \( k_{\Delta L=0} \) and \( k_{\Delta L=\Delta L} \) are the spring constants when the load is applied to the end tip and when it is applied a finite distance \( \Delta L \) (measured along the axis of symmetry of the cantilever) from the end tip, respectively. Note that in Eq. (7), it is assumed that the load is applied along the axis of symmetry of the cantilever, since for a given \( \Delta L \) the variation in the spring constant with off-axis loading is insignificant.

III. RESULTS AND DISCUSSION

We shall first examine in detail the previous PBA formulations, in order to clarify the apparently contradictory reports of Sader et al. and Butt et al.

To begin we note that the PBA is in fact a nonunique approximation, i.e., a number of different formulations are possible. This is due to the fact that the rectangular arms of the V-shaped cantilever are not parallel but skewed to one another. This therefore creates ambiguity in the selection of an appropriate width and length for the two rectangular arms, unless an analysis such as the above is conducted. To our knowledge, only two PBA formulations have appeared in the literature, namely that of Albrecht et al. and Butt et al.

In reference to Table I of Ref. 2 and Figs. 1 and 2 of the present paper, it is evident that Albrecht et al. selected the width and length of their PBA formulation to be \( \bar{d} \) and \( \bar{L} \), respectively. It can be easily shown that their PBA formulation results in the following expression for the spring constant:

\[
\text{FIG. 4. Plot of } \frac{k_{\text{FE}}}{k_{\text{PBA}}} (k \text{ is the spring constant where "FE" and "PBA" refer to results obtained by the finite element method and PBA, respectively, for } \nu=0.25. \text{ Solid line [new PBA, Eq. (6)], long-dashed line [Albrecht PBA, Eq. (8)], short-dashed line [Butt PBA, Eq. (9)] for aspect ratio } A(=b/L) \text{ of (a) } A=0.75, \text{ (b) } A=1, \text{ (c) } A=1.25.}
\]
TABLE I. Summary of formulas presented for the spring constant of V-shaped AFM cantilevers and percentage relative errors evaluated for the typical practical values of $A = 1$, $dlb = 0.2$, and $v = 0.25$.

<table>
<thead>
<tr>
<th>Formulation</th>
<th>Spring constant formula</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Albrecht PBA [Eq. (8)]</td>
<td>$k_{Albrecht} = \frac{E t^3 d}{2 L^3} \left[1 + \frac{b^2}{4L^2}\right]^{-2}$</td>
<td>25%</td>
</tr>
<tr>
<td>Butt PBA [Eq. (9)]</td>
<td>$k_{Butt} = \frac{E t^3 d}{2 L^3} \left[1 + \frac{4d^2}{b^2}\right]^{-1}$</td>
<td>16%</td>
</tr>
<tr>
<td>Zeroth-order solution [Eq. (10)]</td>
<td>$k_0 = \frac{E t^3 d}{2 L^3} \left[1 + \frac{4d^2}{b^2}\right]^{-1}$</td>
<td>13%</td>
</tr>
<tr>
<td>New PBA [Eq. (6)]</td>
<td>$k = \frac{E t^3 d}{2 L^3} \cos \theta \times \left[1 + \frac{4d^2}{b^2} \left(3 \cos \theta - 2\right)\right]^{-1}$</td>
<td>2%</td>
</tr>
</tbody>
</table>

In contrast, Butt et al.\(^5\) selected $d$ and $L$ to be the width and length of the rectangular arms, respectively, and their formulation can be shown to result in the following expression for the spring constant;

$$k_{Butt} = \frac{E t^3 d}{2 L^3}.$$ (8)

Clearly for an aspect ratio $A (= b/L)$ of unity, there exists a discrepancy of $\sim 60\%$ between Eqs. (8) and (9), thus demonstrating the nonuniqueness of the PBA formulation. At this stage we note that the PBA is unique and exact only when the rectangular arms are parallel and infinitely narrow, i.e., when $A \to 0$ and $dlb \to 0$; it is clear that in this limiting case, Eqs. (6), (8), and (9) are identical, as required.

The accuracy of Eq. (8) was examined in detail in Ref. 4, where it was demonstrated that for practical cantilever dimensions, Eq. (8) is in error by $\sim 25\%$. The benchmark for that investigation was the results of a rigorous finite element analysis\(^9\) of the V-shaped cantilever. Butt et al.\(^5\) examined the validity of Eq. (9) and claimed that it was accurate to $\sim 2\%$. However, upon comparison of Refs. 4 and 3, it becomes evident that the benchmark taken by Butt et al.\(^5\) for the spring constant of the V-shaped cantilever was in fact identical to the zeroth-order solution given in Eq. (A5) of Ref. 4 (which completely neglects the anticlastic curvature in the deflection function of the cantilever), namely,

$$k_0 = \frac{E t^3 d}{2 L^3} \left[1 + \frac{4d^2}{b^2}\right]^{-1}.$$ (10)

It is clear from Eqs. (9) and (10) that for a practical value of $dlb \sim 0.2$, Eq. (9) overestimates the zeroth-order solution [Eq. (10)] by $\sim 3\%$, in line with the value obtained by Butt et al.\(^5\) However, the zeroth-order solution itself overestimates the true spring constant, obtained via a rigorous finite element analysis, by $\sim 13\%$.\(^4\) Therefore, the PBA formulation presented in Eq. (9) is not accurate to $\sim 2\%$ as claimed by Butt et al.\(^5\) but is in error by $\sim 16\%$. Therefore, the PBA formulations of Albrecht et al.\(^2\) and Butt et al.\(^5\) are rather poor and inaccurate approximations to the spring constants of V-shaped cantilevers, since they possess typical errors of $\sim 25\%$ and $\sim 16\%$, respectively, for practical cantilever dimensions.

We shall now present a comparison of the PBA formulations of Albrecht et al.\(^2\) Butt et al.\(^5\) and the new PBA formulation Eq. (6), for varying cantilever dimensions. As in Ref. 4, the benchmark taken for this comparison is the results obtained by a rigorous finite element analysis\(^9\) of the governing plate equation. Since practical V-shaped AFM cantilevers have an aspect ratio $A$ of unity and a Poisson's ratio of $v = 0.25$, we shall only present results for $0.75 \leq A \leq 1.25$ and $v = 0.25$. Plots of the ratios of the PBA results to the results of the finite element analysis are presented in Figs. 4(a)–4(c), for varying cantilever dimensions. Note that the new PBA formula, Eq. (6), presents a remarkable improvement in accuracy compared to the previous formulations.
over the previous PBA formulas, Eqs. (8) and (9), while maintaining similar analytical simplicity. For ease of comparison, a summary of the formulas presented in this paper for the spring constant of V-shaped AFM cantilevers and their accuracies for the typical practical values of $A = 1$, $d/b = 0.2$, and $v = 0.25$, is given in Table I. From the table, we see that for these typical values, the Albrecht PBA, Eq. (8), is in error by ~25%, the Butt PBA Eq. (9) is in error by ~16%, whereas the new PBA formula Eq. (6) is only in error by ~2%. We note that the analytical solution presented in Ref. 1 is accurate to ~0.2% for the above typical values, and is therefore considerably more accurate than the new PBA formulation. Nonetheless, we believe the new PBA formula, Eq. (6), will be of considerable practical value, since at present the material and geometric parameters of AFM cantilevers cannot be measured with such accuracy. Also, the result presented in Ref. 1 exhibits greater complexity than the new PBA formulation, as mentioned above.

Finally, we consider the simple formula presented in Eq. (7) for the off-end loading of the V-shaped cantilever. In Figs. 5(a)–5(c) we present a comparison of the results obtained by Eq. (7) to those obtained by a rigorous finite element solution of the governing plate equation, for various cantilever dimensions which encompass cases of practical interest. As can be seen from Figs. 5(a)–5(c), there is good agreement between Eq. (7) and the rigorous finite element solution, with errors only of the order of a few percent, thus verifying that the simple analytical result for the off-end loading of a rectangular cantilever is also applicable to the V-shaped cantilever.

Provided the width and length of the rectangular arms are chosen appropriately, we have shown that the PBA is a valid and accurate approach to the evaluation of the spring constant of V-shaped AFM cantilevers. Due to their accuracy and simplicity, we therefore believe that the new PBA formula, Eq. (6), and its complimentary expression, Eq. (7), for the case of off-end loading will be of considerable practical value to the users of the AFM.

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APPENDIX

In this Appendix we show that, to leading order, the variation in the spring constant of a V-shaped cantilever with load position is in fact identical to that of a rectangular cantilever.

To begin, we apply the zeroth-order method to the equivalent cantilever presented in Fig. 3. In doing so we find that, to leading order, the variation in the spring constant of the V-shaped cantilever with load position is

$$K_1 = \frac{dL = 0}{dL = \Delta L} = \left[1 + \frac{4d^3}{b^3} \frac{1}{3} (3\beta - 2) - \frac{3\Delta L}{L} \left[1 + \frac{4d^2}{b^2} \frac{1}{3} (2\beta - 1) + \frac{3\Delta L^2}{L^2} \times \left(1 - \frac{d}{b} - 2 - \beta \ln \left(2 \frac{d}{b} \frac{L}{\Delta L} \right)\right)\right]\right] \times \left[1 + \frac{4d^3}{b^3} (3\beta - 2) \right]^{-1}. \quad (A1)$$

where $\beta = \cos \theta$ and $\Delta L$ is the distance of the load position from the end tip. Note that in obtaining Eq. (A1), it has been assumed that the load position lies on the axis of symmetry of the cantilever. This is because the application of an off-axis load only produces a small deviation from the on-axis value (~1%), due to the excellent lateral stability of the V-shaped cantilever.

Equation (A1) may be greatly simplified by noting that practical V-shaped AFM cantilevers have the properties

$$d \ll 1, \quad \frac{\Delta L}{L} \ll 1, \quad \beta \ll 1, \quad (A2)$$

from which it is clear that the leading-order behavior of Eq. (A1) is given by

$$K_1 = 1 + 3 \frac{\Delta L}{L} + 3 \left(\frac{\Delta L}{L}\right)^2. \quad (A3)$$

Using Eq. (A2), it is then clear that Eq. (A3) may be very well approximated by

$$K_1 = \left(1 - \frac{L - \Delta L}{L}\right)^3, \quad (A4)$$

which is in fact identical to the result for the rectangular cantilever. Therefore, we have shown that, to leading order, the variation in the spring constant of a V-shaped cantilever with load position is in fact identical to that of a rectangular cantilever.

9. PAFEC is a trademark of, and is available from, PAFEC Ltd, Strelley Hall, Main Street, Strelley, Nottingham NG8 6PE, England.
10. P. Mulvaney (personal communication).