Surface stress induced deflections of cantilever plates with applications to the atomic force microscope: Rectangular plates

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Surface stress is a material property that underpins many physical processes, such as the formation of self-assembled monolayers and the deposition of metal coatings. Due to its extreme sensitivity, atomic force microscopy (AFM) has recently emerged as an important tool in the measurement of surface stress. Fundamental to this application is theoretical knowledge of the effects of surface stress on the deflections of AFM cantilever plates. In this article, a detailed theoretical study of the effects of surface stress on the deflections of rectangular AFM cantilever plates is given. This incorporates the presentation of rigorous finite element results and approximate analytical formulas, together with a discussion of their limitations and accuracies. In so doing, we assess the validity of Stoney’s equation, which is commonly used to predict the deflections of these cantilevers, and present new analytical formulas that greatly improve upon its accuracy. © 2001 American Institute of Physics. [DOI: 10.1063/1.1342018]

I. INTRODUCTION

Surface stress is one of the most widely studied and important properties of a solid surface.1–3 A knowledge of its origin and behavior is fundamental to many industrial processes including the epitaxial growth of semiconductor films,4 deposition of metal coatings,5,6 and the formation of self-assembled monolayers.7 Consequently, a great deal of effort has been focused on understanding the origins and effects of surface stress, using both theoretical first principles calculations,8,9 and experimental techniques.10–15 One of the best known and most commonly used experimental techniques to measure surface stress is the so-called “bending-plate” method. This method uses the property that a differential interfacial stress applied to the faces of an elastic plate will cause the plate to deform. This technique was first proposed by Stoney in 190916 to measure the residual stresses in metallic thin films deposited by electrolysis. In this method, the surface stress is calculated from the observed deformation of the plate using the following simple formula, which is commonly referred to as Stoney’s equation:

\[ \sigma_s = \frac{E t^2}{6(1 - \nu)R}, \]

where \( \sigma_s \) and \( \sigma_s' \) are the surface stresses on the upper and lower surfaces of the plate, respectively, \( E \) is the Young’s modulus of the plate, \( \nu \) is Poisson’s ratio, and \( R \) is its radius of curvature. Equation (1) is applicable to plates of uniform thickness \( t \) that are composed of homogeneous isotropic materials.

Most recently the bending-plate technique has emerged in the field of atomic force microscopy (AFM).7,17–19 These developments have been motivated primarily by the highly desirable properties of AFM cantilevers. In particular, these cantilevers have very small dimensions and high resonant frequencies, which enables very fast and sensitive measurements of surface stress. Stoney’s equation is typically used to convert the observed deflection of the cantilever to a surface stress. However, inherent in Stoney’s equation is the fundamental assumption that the plate bends with uniform curvature, and as such has been derived formally for plates that are unrestrained along their edges. Unfortunately, AFM cantilever plates do not satisfy this condition, since by necessity, at least one of the edges of the plate must be clamped rigidly.

To examine the validity of Stoney’s equation and consequently place this application of atomic force microscopy on a firm footing, a detailed analysis of the effects of surface stress on the deformation of AFM cantilever plates is required.

To date, the effects of surface stress on the deformation of elastic plates have most commonly been analyzed using approximate “strength-of-materials” approaches.20,21 However, such methods inherently neglect the boundary conditions at the edges of the plates, which are of paramount importance to the present study. Consequently, to overcome these difficulties we will implement a complete field description of the plate deformations. Since an exact analytical solution for a cantilever plate is extremely difficult if not impossible to obtain, we shall use two alternative yet complementary approaches to analyze the effects of surface stress. First, we shall use the finite element method to obtain a detailed numerical solution of the governing plate equation. These numerical results will then be used to assess the validity and accuracy of Stoney’s equation. The second approach will involve the development of approximate analytical formulas, which will replace Stoney’s equation in situations where it is found to be inaccurate. Rather than use the governing plate equation directly, we shall implement an energy minimization analysis. This will follow and extend the analysis of Reissner et al.,22 to the case of interest in this
study. Cantilever plates with rectangular and V-shaped plan views shall be considered, since these are the most common cantilevers encountered in practice. Results for rectangular plates are presented in this article (part I), whereas those for V-shaped cantilevers will be given in a companion article (part II). The results and discussion presented in these articles are expected to form the cornerstone for future implementations of atomic force microscopy in the measurement of surface stress.

We begin with a discussion of the background theory and assumptions implemented in Sec. II. This will be followed in Sec. III by a detailed examination of the relationship between Stoney’s equation and the deformation of a cantilever plate. In Sec. IV the principal analytical formulas will be presented. An exact analytical solution for the average deflection function is given in Appendix A, which will be used to validate the analytical formulas of Sec. IV. Finally, results of the finite element analysis, and a detailed assessment of Stoney’s equation and all formulas derived in this article, will be given in Sec. V.

II. BACKGROUND THEORY

In this section we present the background theory for calculating the effects of surface stress on the deflection of a cantilever plate composed of a homogeneous isotropic material. This theoretical framework is applicable to thin plates of arbitrary plan view and uniform thickness exhibiting small deflections, where the effects of in-plane loading on the transverse (out-of-plane) deflections are negligible. Throughout we assume that the applied surface stress is homogeneous and isotropic in nature. A schematic depiction of the problem under consideration is given in Fig. 1.

For a thin plate exhibiting small deflections, the strains in the plate are given by

$$
e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial^2 w}{\partial x^2}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2}, \quad \epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$$

where $u, v, w$ are the displacements of the midplane of the plate in the $x, y, z$ directions, and $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy}$ are $xx, yy, xy$ components of the strain tensor, respectively. When the plate is loaded by homogeneous and isotropic surface stresses, Hooke’s law becomes

$$\sigma_{xx} = \frac{E}{1-\nu} (\epsilon_{xx} + \nu \epsilon_{yy}) + \sigma_s(z),$$

$$\sigma_{yy} = \frac{E}{1-\nu} (\epsilon_{yy} + \nu \epsilon_{xx}) + \sigma_s(z),$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)} \epsilon_{xy},$$

where $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ are the $xx, yy, xy$ components of the stress tensor, respectively, and

$$\sigma_s(z) = \sigma_s^+ \delta(z - \frac{t}{2}) + \sigma_s^- \delta(z + \frac{t}{2}),$$

where $\sigma_s^+$ and $\sigma_s^-$ are the applied surface stresses at the upper and lower faces of the plate (see Fig. 1), and $\delta$ is the Dirac delta function.

Since the effects of in-plane loading on the transverse deflections are neglected and the deflections themselves are small, it follows that in-plane and out-of-plane deformations are uncoupled, i.e., $w$ is independent of $u$ and $v$. We shall not consider the in-plane deformations any further, because they are not normally measured in practice. The transverse deflection function $w$ can be calculated by first evaluating the bending moments per unit length acting on the plate. From Eq. (3) we obtain

$$M_{xx} = \int_{-t/2}^{t/2} \sigma_{xx} z dz = -D \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \Delta \sigma_s \frac{t}{2},$$

$$M_{yy} = \int_{-t/2}^{t/2} \sigma_{yy} z dz = -D \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2} + \Delta \sigma_s \frac{t}{2},$$

$$M_{xy} = \int_{-t/2}^{t/2} \sigma_{xy} z dz = -D(1-\nu) \frac{\partial^2 w}{\partial x \partial y},$$

where $M_{xx}, M_{yy}, M_{xy}$ are the $xx, yy, xy$ components of the moment tensor, respectively, $\Delta \sigma_s = \sigma_s^+ - \sigma_s^-$ is the applied differential surface stress, and $D = E t^3 / (12(1-\nu^2))$.

Satisfaction of the equilibrium conditions for the bending moments and transverse shearing forces (which are evaluated from the bending moments) acting on an infinitesimal element of the plate, then gives the required governing equation

$$\nabla^4 w = 0.$$

We now examine the boundary conditions for Eq. (6) that are appropriate for a cantilever plate. At the clamped edge the plate exhibits no deflection or rotation about any axis, from which it follows that

$$w = \nabla \cdot \mathbf{w}_{\text{clamped edge}} = 0,$$

where $n$ is the local coordinate normal to the edge. At all free edges, the normal components of the bending moments and
III. STONEY’S EQUATION AND THE CANTILEVER PLATE

In this section, we use the above theoretical framework to examine the relationship between Stoney’s equation and the deflection function of a cantilever plate. To begin, we note that the exact deflection function \( w_{\text{free}} \) for a plate that is completely unrestrained, i.e., all edges are free, can be obtained trivially from Eqs. (6) and (8):

\[
w_{\text{free}}(x,y) = \Omega(x^2 + y^2),
\]

where

\[
\Omega = \frac{\Delta \sigma_s t}{4D(1+\nu)}.
\]

Equation (10) is identical to Eq. (1) which is commonly referred to as Stoney’s equation. It predicts that an unrestrained plate will bend with uniform and constant curvature \( 2\Omega \), a result that is independent of the plan view geometry of the plate.

In contrast to the unrestrained (free) plate problem, the solution for a cantilever plate poses a formidable challenge. Indeed, if Eq. (10) is used to predict the deflection function of a cantilever plate, as is commonly performed in practice, the clamp boundary condition at \( x = 0 \) will be violated. Consequently, to accurately calculate the deformation of a cantilever plate due to an applied differential surface stress, a rigorous account of effects due to the clamp must be included. To examine these effects we decompose the cantilever problem into two subproblems:

Subproblem (1): deformation of an unrestrained plate due to an applied differential surface stress, for which Stoney’s equation is the exact solution;

Subproblem (2): deformation of a cantilever plate due to a specified displacement at its clamped end, \( w(0,y) = -\Omega y^2, \partial w/\partial x \big|_{x=0} = 0 \).

A schematic illustration of this decomposition is given in Fig. 3. Subproblem (2) shall henceforth be referred to as the “correction problem,” since for a cantilever plate it accounts for effects omitted in Stoney’s equation.

We emphasize that the above decomposition is exact, and that superposition of subproblems (1) and (2) will give the deflection function \( w_{\text{cant}} \) of a cantilever plate due to an applied surface stress, i.e.,

\[
w_{\text{cant}} = w_{\text{free}} + w_{\text{corr}},
\]

where \( w_{\text{free}} \) and \( w_{\text{corr}} \) are the deflection functions for the free plate and correction problems, respectively. The subscripts cant, free, and corr shall henceforth be used to distinguish between the three problems.

It is clear from Fig. 3 that Stoney’s equation describes the surface stress-induced deformation of a cantilever plate only if (i) the length of the plate greatly exceeds its width, and (ii) the point under consideration is far from the clamp. Indeed, this conclusion provides the rationale for the use of Stoney’s equation in current practice. In the following sections, we shall elucidate the regime of validity of this approximation by analyzing rigorously the effects of surface stress on the deformation of a rectangular cantilever plate.
This will involve the presentation of accurate numerical results and the derivation of improved analytical formulas which utilize the above decomposition.

IV. ANALYTICAL FORMULAS

In this section we present explicit analytical formulas for the static deflection of a rectangular cantilever plate due to an applied surface stress (see Fig. 4). This will consist of asymptotic formulas for plates with large and small plate aspect ratios. Furthermore, an exact analytical solution for the average deflection function when Poisson’s ratio is zero is presented in Appendix A. This latter solution will be used in the following sections to check the validity of asymptotic formulas for the true deflection function \( w_{\text{cant}}(x,y) \).

A. Asymptotic formulas

We now present asymptotic expressions for the deflection function of the plate \( w_{\text{cant}}(x,y) \), for the limiting cases \( L/b \to 0 \) and \( L/b \to \infty \).

When \( L/b \to \infty \), the clamp exerts no effect on the deflection function of the plate in regions away from the clamp, as discussed above. Consequently, the exact expression for the deflection function away from the clamp is given by Stoney’s equation:

\[
w_{\text{cant}}(x,y) = \Omega (x^2 + y^2), \quad \frac{L}{b} \to \infty.
\]  

In contrast, for \( L/b \to 0 \) the clamp completely restricts the deflection function to be independent of \( y \); this is provided the region of the plate under consideration is away from the sides \( y = \pm b/2 \). It then follows from Eqs. (6)–(8) that the deflection function away from these sides is given by

\[
w_{\text{cant}}(x,y) = \Omega (1 + \nu)x^2, \quad \frac{L}{b} \to 0.
\]  

Next we derive formulas that (i) improve on the above asymptotic expressions, and (ii) correct Eqs. (13) and (14) for regions of the plate where they are not valid, i.e., in regions near \( x = 0 \) for Eq. (13), and near \( y = \pm b/2 \) for Eq. (14).

B. Improved asymptotic formula \( (L/b \gg 1) \)

Here we focus on cantilever plates possessing large but finite aspect ratios, i.e., \( L/b \gg 1 \), and derive an analytical expression that improves on Stoney’s equation. This will be accomplished by solving the correction problem using a modification of the method of Reissner et al., which is based on an energy minimization principle.

To begin, the deflection function \( w_{\text{corr}}(x,y) \) for the correction problem is expanded as a power series in \( y \). The first two nonzero terms in this expansion are then retained as an approximation to the true deflection function.\(^{27}\) Since \( w_{\text{corr}} \) is symmetric about the \( x \) axis (see Fig. 4), this gives

\[
w_{\text{corr}}(x,y) \sim f(x) + y^2 \cdot g(x),
\]

where \( f \) and \( g \) are functions purely in terms of \( x \).

To evaluate the functions \( f \) and \( g \), the total potential energy of the correction problem is required. Given that the only load applied to the plate is a static deflection imposed at the clamp, i.e., \( w_{\text{corr}}(0,y) = -\Omega y^2 \), it follows that the total potential energy of the plate is given by its strain energy. The governing differential equations for \( f \) and \( g \) are then evaluated by substituting Eq. (15) into Eq. (9) and minimizing the resulting expression using the calculus of variations. Following this procedure we obtain

\[
\frac{b^4}{180} \frac{d^4 g}{dx^4} - \frac{2b^2(1-\nu)}{3} \frac{d^2 g}{dx^2} + 4(1-\nu^2)g = 0,
\]  

\[
\frac{d^2 f}{dx^2} = \frac{b^2}{12} \frac{d^2 g}{dx^2} + 2\nu g,
\]

with boundary conditions

\[
f = \frac{df}{dx} = \frac{dg}{dx} \bigg|_{x=0} = 0,
\]  

\[
g(0) = -\Omega,
\]  

\[
\left. \frac{d^2 f}{dx^2} + \frac{3b^2}{20} \frac{d^2 g}{dx^2} + 2\nu g \right|_{x=L} = 0.
\]  

\( \Omega \) and \( \nu \) are the plate stiffness and Poisson’s ratio respectively.
\[ \frac{d^3f}{dx^3} + \frac{3b^2}{20} \frac{d^3g}{dx^5} - 2(4 - 5\nu) \frac{dw_2}{dx} = 0. \] (17d)

From Eqs. (16) it is evident that \( f \) and \( g \) contain exponentially growing and decaying functions of \( x \). Since \( L/b \gg 1 \), however, all exponentially growing functions of \( x \) may be neglected, from which we obtain

\[ f(x) = -\Omega b^2 \left\{ \frac{1}{12} + 2\nu \left[ \frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_1 \tau_2} \right] - \left[ \frac{1}{\tau_1} + \frac{1}{\tau_2} \right] \frac{x}{b} \right\} \]
\[ - \frac{2}{\tau_1} \sum_{i=1}^{2} d_i \left\{ \frac{1}{12} + \frac{2\nu}{\tau_i^2} \exp \left[ -\tau_i \frac{x}{b} \right] \right\}, \] (18a)
\[ g(x) = -\frac{\Omega}{2} \sum_{i=1}^{2} d_i \exp \left[ -\tau_i \frac{x}{b} \right], \] (18b)

where

\[ \tau_i = 2\sqrt{3} \left[ \frac{5(1 - \nu) + (-)^i \sqrt{10(1 - \nu)(2 - 3\nu)}}{2} \right]^{1/2}, \] (19a)
\[ d_i = \frac{\tau_{3-i}}{\tau_{3-i} - \tau_i}. \] (19b)

The required solution for the deflection function \( w_{\text{cant}} \) of a cantilever plate due to an applied surface stress is then obtained by substituting Eqs. (15) and (18) into Eq. (12):

\[ w_{\text{cant}}(X,Y) = \Omega L^2 \left\{ X^3 + 2\nu X^2 \left[ \frac{1}{\tau_1} + \frac{1}{\tau_2} \right] \frac{b}{L} \right\} \]
\[ - \left[ \frac{1}{12} + 2\nu \left[ \frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_1 \tau_2} \right] \right] \frac{b}{L} \]
\[ - \frac{2}{\tau_1} \sum_{i=1}^{2} d_i \left\{ \frac{1}{12} + \frac{2\nu}{\tau_i^2} \exp \left[ -\tau_i X \frac{L}{b} \right] \right\} \left\{ \frac{b}{L} \right\}^2 \]
\[ + Y^2 \left[ 1 - \frac{2}{\tau_1} \sum_{i=1}^{2} d_i \exp \left[ -\tau_i X \frac{L}{b} \right] \right]. \] (20)

where \( X = x/L \) and \( Y = y/L \). Note that this expression satisfies the boundary conditions at \( x = 0 \), and gives Stoney’s equation in the limit as \( L/b \rightarrow \infty \), as required. In addition, the exact expression for the average deflection function \( \bar{w} \) [Eq. (A4)] is recovered from Eq. (20) when Poisson’s ratio is zero, thus verifying its validity.

C. Improved asymptotic formula \((L/b \ll 1)\)

For the opposite limit, where \( L/b \ll 1 \), it is appropriate to expand the deflection function \( w_{\text{corr}} \) for the correction problem as a power series in \( x \). Retaining only the first two nonzero terms in the expansion then gives

\[ w_{\text{corr}}(x,y) \sim -\Omega y^2 + x^2 k(y). \] (21)

where the function \( k \) depends only on \( y \), and use has been made of the specified displacement and slope conditions at the clamped end \( x = 0 \). To determine the governing equation for \( k(y) \) we again perform a calculus of variations on the total potential energy of the plate, from which we obtain

\[ \frac{d^4k}{dy^4} + \frac{20}{3L^2} (2 - 3\nu) \frac{d^2k}{dy^2} + \frac{20}{L^2} k = \frac{20}{L^2} \Omega, \] (22)

with

\[ \frac{dk}{dy} = \frac{d^3k}{dy^3} \rfloor_{y=0} = 0, \] (23a)
\[ \frac{d^2k}{dy^2} + \frac{10\nu}{3L^2} k \rfloor_{y=b/2} = \frac{10\Omega}{3L^2}, \] (23b)
\[ \frac{d^3k}{dy^3} \left[ 8 - 10\nu \right] \frac{dk}{dy} \rfloor_{y=b/2} = 0, \] (23c)

where the symmetry properties of the deflection function have been invoked. Solving this governing equation and substituting the result into Eq. (21) and subsequently Eq. (12), then gives the required expression for the deflection function \( w_{\text{corr}} \) of a cantilever plate due to an applied surface stress

\[ w_{\text{corr}}(x,y) = \Omega (1 + \nu) x^2 \left\{ 1 + \sum_{i=1}^{2} g_i \cosh \left[ \gamma_i \frac{y}{L} \right] \right\}, \] (24)

\[ L/b \ll 1. \]

where

\[ \gamma_i = \left[ \frac{10(2 - 3\nu) + (-)^i \sqrt{10(11 - 60\nu + 45\nu^2)}}{3} \right]^{1/2}, \] (25a)
\[ g_i = (-)^i \frac{10(1 - \nu)}{3H} \left[ \frac{20}{\gamma_{3-i}} + \frac{10\nu}{3} \gamma_{3-i} \right] \sinh \left[ \gamma_{3-i} \frac{b}{2L} \right], \] (25b)
\[ H = \sum_{i=1}^{2} (-)^i \left[ \gamma_i^2 + \frac{10\nu}{3} \right] \left[ \frac{20}{\gamma_{3-i}} + \frac{10\nu}{3} \gamma_{3-i} \right] \]
\[ \times \sinh \left[ \gamma_{3-i} \frac{b}{2L} \right] \cosh \left[ \gamma_{3-i} \frac{b}{2L} \right]. \] (25c)

As for the \( L/b \gg 1 \) asymptotic formula, we find that (i) Eq. (24) satisfies the exact solution for the average deflection function when \( \nu = 0 \), and (ii) the corresponding limiting form Eq. (14) is recovered when \( L/b \rightarrow 0 \).

Finally, we note that higher order terms in \( x \) can be included in Eq. (21) for a more accurate representation of the true deflection function. However, it was found that this greatly increased the complexity of the solution, and consequently was not pursued.

V. RESULTS AND DISCUSSION

The deflection properties of the rectangular cantilever plate shall now be examined. This will include a detailed assessment of the validity of the above asymptotic formulas and presentation of accurate numerical solutions of Eq. (6). These latter results are obtained using a finite element (FE) analysis.29
A. Qualitative behavior of deflection function

To begin we study the qualitative influence of the clamp on the deflection function of the plate. Cases for $L/b > 1$ and $L/b < 1$ are considered separately, and results presented for the differences between the limiting asymptotic formulas [Eqs. (13) and (14)], and the true deflection function; for $L/b > 1$ results for the correction problem are given, whereas for $L/b < 1$ the difference between the limiting asymptotic formula Eq. (14) and the true deflection function are presented.

First, we consider cantilever plates with $L/b > 1$, which is the case encountered most often in practice. In Fig. 5, FE results for the correction problem are presented for $L/b = 2$ and various Poisson’s ratios $\nu$; results for larger aspect ratios $L/b$ are qualitatively identical to Fig. 5. These results demonstrate clearly that curvature is affected only in a small region near the clamp at $x=0$. Away from the clamp, the plate undergoes a rigid-body displacement and rotation, which is strongly dependent on Poisson’s ratio. Note in Figs. 5(a)–5(c) that the scaled displacements at the clamped end ($x=0$) are identical.

These findings are to be compared against the predictions of the asymptotic formula for the correction problem presented in Eqs. (15) and (18). From Eq. (18) it is clear that a thin boundary layer of width $O(b)$ exists near the clamp, where the curvature of the plate is nonzero; these curvature effects decay exponentially with distance from the clamp. Outside this boundary layer, however, Eq. (18) predicts that the plate undergoes a rigid-body displacement and rotation, which is dependent on Poisson’s ratio $\nu$, for $\nu = 0$ the plate undergoes a pure rigid-body displacement of magnitude $\Omega b^2/12$, whereas for nonzero $\nu$ an additional displacement and rotation is superimposed. These analytical predictions are in complete agreement with the numerical calculations presented in Fig. 5, and demonstrate that the clamp has a significant effect on the curvature at the tip $x = L$, but strongly affects the tip displacement and slope when $L/b > 1$. This finding applies directly to $w_{\text{corr}}$, since $w_{\text{corr}}$ is given by the sum of $w_{\text{corr}}$ and Stoney’s equation $w_{\text{free}}$. It should also be noted that the predictions of Eqs. (15) and (18) are qualitatively indistinguishable from the FE results presented in Fig. 5, but small quantitative differences do exist. The magnitude of these quantitative errors shall be investigated in detail below.

Next we examine the complementary case where $L/b < 1$. In Fig. 6 we present FE results for the difference between $w_{\text{corr}}$ and the limiting asymptotic formula [Eq. (14)]. Results for $L/b = 0.2$ are presented only, because the behavior of the deflection function for other aspect ratios is qualitatively similar. The scaling of the axes in Figs. 6(a)–6(c) are identical to facilitate comparison. From Fig. 6 it is evident that the largest difference between the true deflection function and the limiting asymptotic formula Eq. (14) occurs near the free edges at $y = \pm b/2$. This observation is in line with the discussion in Sec. III, where it was predicted that Eq. (14) will not be valid near $y = \pm b/2$. Nonetheless, these edge effects decay rapidly away from $y = \pm b/2$, in agreement with Eq. (24), which predicts that they decay exponentially fast.

Corresponding analytical results obtained by taking the difference between Eqs. (14) and (24) are given in Fig. 7. Comparison of Figs. 6 and 7 reveals that the predictions of Eq. (24) are similar to those of the FE analysis, but the deflection function at $y = \pm b/2$ is not described well. Equation (24) predicts a quadratic dependence on $x$ whereas the true behavior is more complicated. Such a discrepancy is not unexpected given that the deflection function in Eq. (24) is artificially restrained to vary quadratically with $x$. We will have more to say about the accuracy and validity of Eq. (24).

B. Quantitative behavior of end tip

We now present numerical results for the behavior of the end tip of the plate ($x = L, y = 0$). This point on the cantilever is chosen because it is commonly measured in practice.
Figure 8 presents results obtained using the FE analysis for the displacement, slope, and curvature. The number of elements were refined systematically to ensure an accuracy of better than 0.1%. These results cover a wide range of aspect ratios $L/b$ and Poisson's ratios $\nu$, and are therefore expected to be of value in practical surface stress measurements.

The results presented in Fig. 8 can also be used to assess the accuracy of Stoney's equation, Eq. (10), since they have been scaled by its predictions. Note that for large aspect ratio ($L/b = 10$), Stoney's equation describes accurately the be-

FIG. 6. Difference between FE solution and $L/b \to 0$ limiting asymptotic formula Eq. (14) for the deflection function: FE solution minus Eq. (14). Results given for $L/b = 0.2$. Deflection function $w(X,Y)$ is scaled by $\Omega L^2$; (a) $\nu = 0$, (b) $\nu = 0.25$, (c) $\nu = 0.49$. Scaled coordinates are $X = x/L$, $Y = y/L$, $Z = z/(\Omega L^2)$.

FIG. 7. Difference between $L/b \ll 1$ improved asymptotic formula Eq. (24) and limiting asymptotic formula Eq. (14) for the deflection function: Eq. (24) minus Eq. (14). Results given for $L/b = 0.2$. Deflection function $w(X,Y)$ is scaled by $\Omega L^2$; (a) $\nu = 0$, (b) $\nu = 0.25$, (c) $\nu = 0.49$. Scaled coordinates are $X = x/L$, $Y = y/L$, $Z = z/(\Omega L^2)$.
behavior of the end tip, with errors not exceeding 4% for the displacement and slope, and <0.1% for the curvature. However, as $L/b$ is reduced, the accuracy of Stoney’s equation decreases. This is expected, since Stoney’s equation is derived in the limit as $L/b \to \infty$. The decrease in accuracy is gradual and for $L/b = 1$ the maximum error is $\sim 20\%$ in the displacement and slope, and $\sim 10\%$ in the curvature. Interestingly, we also find that for $L/b > 3$, the predictions of curvature by Stoney’s equation are indistinguishable from those of the FE analysis. These findings demonstrate that Stoney’s equation can be used to produce reasonable results for cantilevers with aspect ratios $L/b > 1$, which is the case most often encountered in practice.

Next we assess the accuracy of the improved asymptotic formula Eq. (20), which is derived in the limit of large but finite aspect ratio $L/b$. A comparison of results obtained using Eq. (20) to those of the FE analysis is given in Fig. 9. These results demonstrate clearly that Eq. (20) provides a remarkable improvement over Stoney’s equation for the displacement and slope when $L/b > 1$; the maximum error for $L/b \geq 1$ is 2% in the displacement and 1% in the slope. This improvement is due to inclusion of the rigid body movement induced by the clamp (see Fig. 5), an effect that is omitted in Stoney’s equation. In contrast, the improvements in curvature are less pronounced, with errors approximately halved. This is of little consequence, however, since as $L/b$ increases from unity, the accuracy improves dramatically, e.g., for $L/b \geq 2$ the error is less than 1% and for $L/b \geq 3$ the predictions of Eq. (20) are indistinguishable from the FE results [see Fig. 9(c)].
Since the curvature effects induced by the clamp decay exponentially fast [see Eq. (20)], they can be omitted when examining the behavior near the end tip for \( L/b > 1 \). This simplification results in the following expression

\[
W_{\text{cant}}(X,Y) = \Omega L^2 \left( X^2 + Y^2 + 2 \nu X \left[ \frac{1}{\tau_1} + \frac{1}{\tau_2} \right] \right) \left( \frac{b}{L} \right) \left( \frac{b}{L} \right)^2 .
\]

(26)

where \( X = x/L \) and \( Y = y/L \). Note that Eq. (26) and Stoney’s equation, Eq. (13), give identical results for the curvature, because all clamp-induced curvature effects are neglected in Eq. (26); this is of no concern since Stoney’s equation predicts accurately the curvature of the end tip. An assessment of the predictions of Eq. (26) for the displacements and slopes is given in Fig. 10. We find that Eqs. (20) and (26) give identical results for \( L/b \geq 2 \), and only differ slightly for \( 1 \leq L/b \leq 2 \). Since Eq. (26) greatly improves upon Stoney’s equation, and the displacement or slope of the end tip are measured typically in practice, we recommend that it be used in place of Stoney’s equation when \( L/b > 1 \). Equation (20) should only be used if improvements in curvature are required.

Although cases when \( L/b < 1 \) are rarely encountered in practice, it is of fundamental interest to investigate the validity of analytical formulas derived for this regime. First, we assess the accuracy of the limiting asymptotic formula, Eq. (14), which is derived in the limit \( L/b \to 0 \). A comparison of the predictions of Eq. (14) to those obtained using the finite element analysis is given in Fig. 11. Note that Eq. (14) gives virtually identical results to the finite element analysis when \( L/b = 0.1 \), but its accuracy decreases as the aspect ratio \( L/b \) increases. Nonetheless, Eq. (14) can be relied upon to give results accurate to within \( \sim 10\% \) provided the aspect ratio \( L/b < 0.4 \). For larger \( L/b \) the error increases dramatically, and for \( L/b = 1 \) the maximum error is 25% in displacement and slope, and 34% in curvature.

The results in Fig. 11 are to be compared against those given in Fig. 12 for the improved asymptotic formula, Eq. (24), which accounts approximately for free edge effects omitted in the limiting asymptotic formula, Eq. (14). Note the significant improvement in accuracy afforded by Eq. (24).
in comparison to the limiting formula Eq. (14). Specifically, Eq. (24) exhibits accuracies better than 10% in the displacement, slope, and curvature, provided \( L/b < 2 \), \( L/b < 1 \), and \( L/b < 0.8 \), respectively. Use of Eq. (24) in cases where \( L/b < 1 \) will therefore guarantee reasonable accuracy (±10%) over the entire regime.

Finally, in Table I we present a summary of the maximum errors exhibited by the five asymptotic formulas Eqs. (13), (14), (24), (20), and (26) considered above, in their respective regimes, i.e., \( L/b \geq 1 \) for Eqs. (13), (20), (26), and \( L/b \leq 1 \) for Eqs. (14) and (24). These results demonstrate clearly that the improved asymptotic formulas [Eqs. (20), (24), and (26)] present a significant improvement over Stoney’s equation, Eq. (13), and the complementary \( L/b \rightarrow 0 \) limiting formula, Eq. (14). Consequently, we recommend use of Eq. (26) when \( L/b \geq 1 \), whereas if \( L/b < 1 \), Eq. (24) should be used.

### VI. CONCLUSIONS

A detailed theoretical study of the effects of surface stress on the deflection properties of rectangular AFM cantilever plates has been presented. This involved a rigorous finite element analysis of the governing plate equation, and the derivation of approximate analytical formulas based on energy minimization techniques.

It was found that the clamp can significantly affect the surface stress-induced deflection of the cantilever plate. In particular, for the typical practical case \( L/b > 1 \), the clamp induces a rigid body displacement and rotation of the end tip, which in turn introduces significant errors in Stoney’s equation. The improved asymptotic formula, Eq. (26), accounts for such effects and gives very good accuracy (<2% in the displacement and slope). Therefore, Eq. (26) should be used in place of Stoney’s equation when \( L/b > 1 \). If improvements in curvature are required then we recommend use of Eq. (20). For the complementary case when \( L/b < 1 \), it was found that the free edge effects neglected in the \( L/b \rightarrow 0 \) limiting formula, Eq. (14), can be important. These effects are included in the improved asymptotic formula, Eq. (24), which presents a significant improvement over Eq. (14).

It is therefore recommended that these new formulas, which are derived for finite aspect ratio \( L/b \), be used in place of (i) Stoney’s equation, which is valid formally for \( L/b \rightarrow \infty \), and (ii) the complementary formula for \( L/b \rightarrow 0 \), Eq. (14). If greater accuracy is required than that afforded by the above analytical formulas, then use should be made of the finite element results provided.

Calculations for AFM cantilevers with V-shaped plan views shall be presented in part II.23

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APPENDIX A

In this Appendix we examine the average deflection of a rectangular cantilever for the special case when Poisson’s ratio \( \nu = 0 \), where an exact solution is possible. The average deflection function \( \bar{w}(x) \) is defined to be

\[
\bar{w}(x) = \frac{1}{b} \int_{-b/2}^{b/2} w_{\text{can}}(x, y) \, dy,
\]

(A1)

where \( b \) is the width of the plate (see Fig. 4). The function \( \bar{w} \) can then be evaluated by integrating Eq. (6) over the width of the plate and making use of the zero transverse shearing force free edge condition, Eq. (8b), for \( \nu = 0 \). Following this procedure then gives

\[
\frac{d^4 \bar{w}}{dx^4} = 0.
\]

(A2)

The corresponding boundary conditions for Eq. (A2) are evaluated in an identical manner, where use is now made of the property that the shear moments \( M_{n} \) are continuous at the free corners of the plate,

\[
\left[ \frac{\bar{w}}{dx} \right]_{x=0} = 0, \quad \left[ \frac{d^2 \bar{w}}{dx^2} \right]_{x=L} = 2\Omega, \quad \left[ \frac{d^3 \bar{w}}{dx^3} \right]_{x=L} = 0,
\]

(A3)

where \( L \) is the length of the plate, \( \Omega \) is defined in Eq. (11) and is evaluated at \( \nu = 0 \). The exact expression for \( \bar{w} \) then directly follows, and is given by

\[
\bar{w}(x) = \Omega x^2.
\]

(A4)

Interestingly, we find that the average deflection function for \( \nu = 0 \) is independent of the aspect ratio \( L/b \). This formula is used in Sec. IV to check the validity of asymptotic expressions for the true deflection function \( w_{\text{can}}(x, y) \).

16Stoney originally considered the case of uniaxial stress, which was later corrected to account for isotropic and homogeneous surface stresses.
22E. Reissner and M. Stein, NACA Tech. Note No. 2369
24Maximum deflections exhibited in practice are typically far smaller than the thickness of the plate.
25From Saint Venant’s principle, a thin boundary layer is expected at the clamped end \( x = 0 \) for \( L/b \gg 1 \). Such an expansion will enable the modeling of this boundary layer.
26From Saint Venant’s principle, thin boundary layers are expected at the free ends \( y = \pm b/2 \) for \( L/b \ll 1 \). Such a power series will enable the modeling of these boundary layers.
27LUSAS is a trademark of, and is available from FEA Ltd. Forge House, 66 High St., Kingston Upon Thames, Surrey KT1 1HN, UK. Quadrilateral thin plate elements with linear interpolation were used throughout.
28From Saint Venant’s principle, thin boundary layers are expected at the free ends \( y = \pm b/2 \) for \( L/b \ll 1 \). Such a power series will enable the modeling of these boundary layers.
29End-tip displacements can be measured with a scanning tunneling microscope tip, whereas the slope can be measured using the optical deflection technique.