Effect of surface stress on the stiffness of cantilever plates: Influence of cantilever geometry

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Numerous measurements have indicated that surface stress can significantly modify the stiffness of cantilever sensors. In contrast, theoretical calculations using classical beam theory predict that stiffness is independent of surface stress. Using a three-dimensional analysis, we recently showed that surface stress does indeed have an effect within the framework of linear elasticity. However, only cantilevers of rectangular geometry were explored. Here, we vary cantilever geometry and find that it plays a critical role, with V-shaped cantilevers displaying greatly enhanced sensitivity in comparison to rectangular cantilevers. Tuning cantilever geometry therefore provides a sensitive route to controlling the effects of surface stress.

Cantilever sensors are capable of detecting environmental changes with extreme sensitivity, examples of which include their use as mass balances, chemical/biological sensors, and viscometers. For mass measurements, it is independent and strain-dependent remains elusive. The precise mechanism giving rise to these effects is only dependent on the adsorbed mass. However, identification of the effects of cantilever geometry on this surface stress effect, and the stiffness of rectangular cantilevers. Recently, we showed that it plays a critical role in this transduction.

The most popular approach to theoretically describe the effects of strain-independent surface stress on cantilever plates, is to use the so-called axial stress model which is incorporated into classical beam theory. However, this model violates force equilibrium and is thus unphysical; correctly used, classical beam theory predicts that surface stress does not affect cantilever stiffness. Recently, we showed that consideration of the full three-dimensional (3D) stress field generated by a surface stress load can indeed affect the stiffness of rectangular cantilevers. Here, we explore the effects of cantilever geometry on this surface stress effect, and find that it plays a critical role in this transduction.

We consider two geometries, which are rectangular and V-shaped cantilevers, since these are most commonly used in practice, see Fig. 1. It is assumed that homogeneous and isotropic surface stresses are applied to top and bottom surfaces, and denoted \( \sigma_s^+ \) and \( \sigma_s^- \), respectively, only the total surface stress \( \sigma_s^T = \sigma_s^+ + \sigma_s^- \) can affect the cantilever stiffness. Importantly, our study focuses on the effect of surface stress change, as is commonly reported. Thus, \( \sigma_s^+ \) and \( \sigma_s^- \) are taken to be the changes in surface stress from their base (intrinsic) values.

To calculate the effect of surface stress on the stiffness of these cantilevers, we use the general approach described in our previous work. This allows for determination of the effect of strain-independent surface stress on the stiffness of cantilever plates of arbitrary geometry, within the framework of linear elasticity. The general approach proceeds by decomposing the surface stress problem into the following subproblems:

Subproblem (1): Deformation of the unrestrained plate under application of a total surface stress \( \sigma_s^T \).

Subproblem (2): Cantilever plate with no surface stress load and a specified in-plane displacement at its clamped base.

FIG. 1. Schematic showing geometry of (a) rectangular and (b) V-shaped cantilevers. The V-shaped cantilever diagonal arm width is \( d = d_1 \sqrt{1 + b^2 / (4 L^2)} \).
The displacement condition in subproblem (2) accounts for violation of the clamp condition in subproblem (1) for a completely unrestrained plate.

As discussed in Ref. 14, superposition of these subproblems gives the required in-plane deformation of the original cantilever problem, with exact satisfaction of free-edge and clamped boundary conditions. This approach can be used to calculate either the change in stiffness, or equivalently, the change in resonant frequency of the cantilever under a surface stress load. We focus on the latter, since this is most commonly reported in practice.

We consider a V-shaped cantilever whose thickness is much smaller than its plan view dimensions. The leading order dependence of the relative change in frequency $\Delta \omega/\omega_0$ on surface stress is then easily determined from the thin plate equation

$$\frac{\Delta \omega}{\omega_0} \sim O\left( \frac{d}{h} \right)^2,$$

where $\omega_0$ is the resonant frequency in the absence of surface stress, $\Delta \omega=\omega-\omega_0$, and $\bar{\sigma}=(1-\nu)\sigma_{yy}/(Eh)$, where $E$ is Young’s modulus and $\nu$ is Poisson’s ratio, cantilever dimensions are specified in Fig. 1. This scaling relation is used in the numerical results below.

We begin by analyzing the V-shaped cantilever, the results of which are then compared to those for the rectangular cantilever.\textsuperscript{14} Due to the complexity involved, solution to subproblem (2) is obtained numerically using a 3D finite element analysis;\textsuperscript{16} subproblem (2) directly gives the change in stiffness of the cantilever.\textsuperscript{14} Results for V-shaped cantilevers with a range of aspect ratios $L/b$ and width ratios $d/h$ are given in Fig. 2.

Comparing these results to those for a rectangular cantilever (Fig. 2 of Ref. 14), we immediately observe that cantilever geometry can greatly influence the effect of surface stress. One major distinction between these two sets of results (for V-shaped and rectangular cantilevers), is that the surface stress effect is opposite in sign. The results in Fig. 2 also verify the scaling dependence of the relative frequency shift on the width ratio $d/h$ as predicted in Eq. (1).

Next, we quantitatively compare the performance of rectangular and V-shaped cantilevers. This comparison is made under the following constraints: both types of cantilevers are composed of the same material, possess identical thickness $h$ and length $L$, and have identical normal stiffness. These constraints ensure that the fundamental resonant frequencies and mass are of similar order, enabling an equitable assessment of geometric effects. Requiring identical normal stiffness then leads to the following relation between the rectangular cantilever width $c$ and the V-shaped cantilever diagonal arm width $\bar{d}$,\textsuperscript{15}

$$c = 2\bar{d} \left( 1 + \frac{4d^3}{b^3} \right)^{-1}.$$  

While approximate in nature, this formula is sufficient for our purposes.

Numerical results for the relative frequency shift of equivalent rectangular and V-shaped cantilevers are then calculated. Linear regression is used to extract the linear component of the relative frequency shift versus surface stress relation, as in Ref. 14, since Eq. (1) is derived in this linear limit using the small deflection theory for thin plates.\textsuperscript{14} We also extrapolate solutions at finite thickness to the zero thickness limit, in accordance with this theory. This procedure is robust since Eq. (1) accurately captures the scaling dependence for large width ratios $d/h$ (Fig. 2); analogous extrapolations are shown in Fig. 3 of Ref. 14 for rectangular cantilevers.

Results of this analysis for rectangular and V-shaped cantilevers are given in Fig. 3. We specifically examine V-shaped cantilevers with an aspect ratio $L/b=1$, since aspect ratios of order unity are commonly encountered in practice. Identical scalings are used for both rectangular and V-shaped cantilevers, enabling a quantitative assessment of their relative performance.

Figure 3 clearly shows that the effect of surface stress is reversed in sign when cantilever geometry is changed from rectangular to V-shape. In addition, we find that the dependence of Poisson’s ratio is dramatically different for rectangular and V-shaped cantilevers. While rectangular cantilevers possess a strong dependence on Poisson’s ratio, varying as $\nu(1-\nu)$, V-shaped cantilevers possess a relatively weak dependence, $(1-\nu)$. Note that the scaled surface stress $\bar{\sigma}$ is inversely proportional to the biaxial modulus, $E/(1-\nu)$, which gives rise to the $(1-\nu)$ dependence for V-shaped cantilevers.
For a typical Poisson’s ratio $\nu=0.25$ and practical width ratios $d/b=0.05–0.3$, we find that V-shaped cantilevers exhibit a relative frequency shift 6–24 times greater than those of equivalent rectangular cantilevers. The factor of 24 corresponds to the lowest width ratio $d/b=0.05$, and decreases monotonically with increasing $d/b$. The relative sensitivity of V-shaped cantilevers also increases as Poisson’s ratio decreases, as is evident from Fig. 3. Collectively, these results show that practical V-shaped cantilevers display greatly enhanced sensitivity to surface stress in comparison to equivalent rectangular cantilevers.

As the aspect ratio $L/b$ is reduced (resulting in a wider cantilever), the relative sensitivity of V-shaped cantilevers is enhanced. For example, given an aspect ratio of $L/b=0.5$, Poisson’s ratio $\nu=0.25$ and $d/b=0.05$, we find that the V-shaped cantilever is 82 times more sensitive to the effects of surface stress than the equivalent rectangular cantilever. Note that a V-shaped cantilever of large aspect ratio, $L/b \gg 1$, is geometrically similar to a rectangular cantilever (i.e., a beam) and hence exhibits comparable sensitivity.

We now explore the physical mechanisms involved in surface stress transduction of both cantilever geometries. For a rectangular cantilever, we demonstrated in Ref. 14 that application of surface stress results in an in-plane deformation of the plate localized at the clamped end, see Fig. 4(a). Far from the clamped end, in-plane strain and hence stress is effectively zero. This immediately leads to the important conclusion that the surface stress effect is diminished in rectangular cantilevers as $L/c$ increases; reducing Poisson’s ratio $\nu$ also decreases the effect.14

The behavior of V-shaped cantilevers is dramatically different. Figure 4(b) provides some insight into this difference, demonstrating that application of surface stress induces nonzero in-plane strain throughout the cantilever, not just at the clamped end. In the immediate vicinity of the clamped end, the arms exhibit an in-plane deformation similar to that of the rectangular cantilever. This is caused by the local strain load at the clamped end of each arm, as specified in subproblem (2). Importantly, this strain load also induces a relative displacement of the two arms of the V-shaped cantilever. This secondary effect gives rise to beam-like deformations of the arms that dominate the aforementioned localized effect at the clamped end; these beam-like deformations are clearly visible in Fig. 4(b). The observed Poisson’s ratio dependence of $(1-\nu)$ for V-shaped cantilevers is thus expected, since (i) the surface stress load at the clamped end in subproblem (2) is proportional to $(1-\nu)$, and (ii) beam-like deformations are independent of Poisson’s ratio $\nu$.

Consider the following practical values for a V-shape cantilever: $L=200 \mu m$, $b=200 \mu m$, $h=0.5 \mu m$, $d=40 \mu m$, $E=179$ GPa, and $\nu=0.25$.17 For a typical surface stress change of $\sigma_s=10^{-3}$ N/m, the above calculations yield a relative frequency shift of $\Delta \omega/\omega_0=2.7 \times 10^{-6}$. This value is much smaller than measurements which typically report frequency shifts of a few percent.3,6,8,10,11 We therefore again conclude14 that reported variations in stiffness that have been attributed to strain-independent surface stress, for a wide range of cantilevers,3,6,8,10,11 are not theoretically described using linear elasticity. This contrasts to clamped-clamped beams, whose surface stress response is accurately described using classical beam theory.

Importantly, a host of measurements report variations in stiffness due to surface modification,2–6,8,10,11 some of which can be predicted using surface elasticity models3 whereas others remain unexplained.2,3,6,8,10,11 Our analysis suggests that these latter experimental studies of cantilevers reporting a stiffness change due to strain-independent surface stress may be due to (i) another mechanism(s), and/or (ii) lie in effects not contained within the standard linear framework.

We have investigated the role of geometry on the transduction of surface stress into changes in cantilever stiffness. Varying the plan view geometry of the cantilever was found to dramatically influence this surface stress effect, with order-of-magnitude enhancements possible. This provides a simple and sensitive route to controlling the influence of strain-independent surface stress, and is expected to find application in (i) experimental studies that seek to elucidate the role of surfaces on cantilever stiffness2–6,8,10,11 and (ii) fine tuning of the resonance properties of practical cantilevers. This is particularly important for nanoscale devices, where such surface based effects are enhanced.

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